

Silver Cubes

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Abstract. An $n \times n$ matrix A is said to be *silver* if, for $i = 1, 2, \dots, n$, each symbol in $\{1, 2, \dots, 2n - 1\}$ appears either in the i th row or the i th column of A . The 38th International Mathematical Olympiad asked whether a silver matrix exists with $n = 1997$. More generally, a *silver cube* is a triple (K_n^d, I, c) where I is a maximum independent set in a Cartesian power of the complete graph K_n , and $c : V(K_n^d) \rightarrow \{1, 2, \dots, d(n - 1) + 1\}$ is a vertex colouring where, for $v \in I$, the closed neighbourhood $N[v]$ sees every colour. Silver cubes are related to codes, dominating sets, and those with n a prime power are also related to finite geometry. We present here algebraic constructions, small examples, and a product construction. The nonexistence of silver cubes for $d = 2$ and some values of n , is proved using bounds from coding theory.

Key words. silver matrix, graph colouring, coding theory, domination in graphs, finite projective geometry

1. Introduction

An $n \times n$ matrix A is said to be *silver* if, for every $i = 1, 2, \dots, n$, each symbol in $\{1, 2, \dots, 2n - 1\}$ appears either in the i th row or the i th column of A . A problem of the 38th International Mathematical Olympiad in 1997 introduced this definition and asked to prove that no silver matrix of order 1997 exists. In [7] the motivation behind this problem as well as a solution is presented: a silver matrix of order n exists if and only if $n = 1$ or n is even.

The *Cartesian product* of two graphs G and H , denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H)$ in which two vertices (x, y) and (x', y') are adjacent if and only if $x = x'$ and $yy' \in E(H)$, or $y = y'$ and $xx' \in E(G)$. An $n \times n$ matrix A can be thought of as a labeling of the vertices of $K_n \square K_n$, where K_n is the complete graph of order n . If $V(K_n) = \{1, 2, \dots, n\}$, then $I = \{(i, i) \mid i = 1, 2, \dots, n\}$ corresponds to the diagonal cells of A . Thus, a silver matrix is a (proper) $(2n - 1)$ -colouring of $K_n \square K_n$ such that for every $x \in I$, every colour appears either on x or on a neighbour of x .

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Definition 1.1 Any maximum independent set of a graph is called a *diagonal* of that graph. Let c be a proper $(r + 1)$ -colouring of an r -regular graph G . A vertex x in G is said to be *rainbow* with respect to c if every colour appears in the *closed neighbourhood* $N[x] = N(x) \cup \{x\}$. Given a diagonal I of G , the colouring c is said to be *silver with respect to I* if every $x \in I$ is rainbow with respect to c . We say G is *silver* if it admits a silver colouring with respect to some diagonal. If all vertices of G are rainbow, then c is called a *totally silver* colouring of G and G is said to be *totally silver*.

With our definition, silver matrices are closely related to *defining sets* of graph colourings (cf. Section 6). This helps justify our definition.

The *Cartesian power* G^d of a graph G is the graph recursively defined by $G^1 = G$, and $G^d = G^{d-1} \square G$ for $d > 1$. The graph K_n^d is called a *cube of side n and dimension d* or an (n, d) -*cube* for short. A silver colouring of an (n, d) -cube (with respect to some diagonal) is called a *silver (n, d) -cube*. In this paper we study silver (n, d) -cubes. These design-like objects exist for only certain pairs (n, d) . Although we know of no existing literature (for $d > 2$), silver (n, d) -cubes are attractive, challenging to construct, and appear to be connected with classical combinatorics, including coding theory, projective geometry and dominating sets in graphs.

We summarize here the main results of this paper.

- (i) If the (m, d) -cube and the (n, d) -cube are silver, then the (mn, d) -cube is silver.
- (ii) For every $n = 2^r 3^s 5^t 7^u$ the $(n, 3)$ -cube is silver.
- (iii) If q is a prime power and $t \geq 1$, then the (q, q^t) -cube is silver and the $(q, \frac{q^t-1}{q-1})$ -cube is totally silver.
- (iv) If p is a prime, then the (p^r, p^s) -cube is silver for every $r, s \geq 0$.
- (v) If $d = 2^t - 1$ or $d = 2^t$ for some t , then the $(2, d)$ -cube is silver. Moreover if d is odd, then every silver $(2, d)$ -cube is totally silver and these exist if and only if $d = 2^t - 1$ for some t . If $k \geq 1$ then no silver $(2, 4k + 2)$ -cube exists.

Claims (i) and (ii) are proved in Section 3. Claims (iii) and (iv) are proved in Sections 4. Claim (v) is proved in Section 5.

2. Notation and motivation

We often identify the vertices of K_n^d with words of length d from either the alphabet $S = \{0, 1, \dots, n-1\}$, or when n is a prime power, with d -dimensional vectors over an n -element field \mathbb{F}_n . In symbols, $V(K_n^d) = S^d$ or $V(K_n^d) = \mathbb{F}_n^d$. We use bold face characters to emphasize that a variable is a word or a vector. The *Hamming distance* of two words is the number of coordinates in which they differ. Two vertices of K_n^d are adjacent if and only if they are at Hamming distance 1. Note that for every $k \in S$, the set

$$I_k = \{\mathbf{x} = x_1 \dots x_d \in S^d \mid x_1 + \dots + x_d = k \pmod{n}\}$$

is a diagonal of K_n^d . We call I_0 the *back-circulant diagonal* of K_n^d . This terminology is motivated by a bijective correspondence between diagonals of K_n^3 and Latin squares of order n . Every diagonal of K_n^d has cardinality n^{d-1} .

We use the following in Section 5.

Proposition 2.1 *For any silver (n, d) -cube with respect to a diagonal I , the number of times each colour appears on I is congruent to n^{d-1} modulo d .*

Proof. Let $c : V(K_n^d) \rightarrow \{0, 1, \dots, d(n-1)\}$ be a silver colouring with respect to I . Let $i \in \{0, 1, \dots, d(n-1)\}$ be a colour. Every $\mathbf{v} \in I$ with $c(\mathbf{v}) \neq i$ has a unique neighbour $\mathbf{x} \notin I$ with $c(\mathbf{x}) = i$. On the other hand every vertex \mathbf{x} of K_n^d not in I with $c(\mathbf{x}) = i$ has exactly d neighbours in I . Therefore the vertices of I which have a colour other than i are partitioned into sets of size d according to their neighbours which have colour i . This implies that d divides $|\{\mathbf{v} \in I \mid c(\mathbf{v}) \neq i\}| = n^{d-1} - |\{\mathbf{v} \in I \mid c(\mathbf{v}) = i\}|$. \square

Once a diagonal I is specified, one can reduce the problem of finding a silver colouring of an r -regular graph G to an ordinary vertex colouring problem as follows. Let H be the graph obtained from G by joining every two vertices which have a common neighbour in I . Then by the definition of a silver colouring, G has a silver colouring with respect to I if and only if $\chi(H) = r + 1$, where $\chi(H)$ denotes the chromatic number of H . On the other hand since I is an independent set in G and the neighbourhood of every $x \in I$ in H is an r -clique, we have $\chi(H) = r + 1$ if and only if $\chi(H - I) \leq r + 1$. We call the graph $H - I$ the *silver reduction* of G with respect to I and denote it by $\text{SR}(G, I)$. This proves the following.

Proposition 2.2 *Let I be a diagonal of an r -regular graph G . Then G has a silver colouring with respect to I if and only if $\text{SR}(G, I)$ is $(r + 1)$ -colourable.*

This reduction is critical in accelerating computer searches for silver cubes. It is also used in Section 5 to prove the nonexistence of certain silver cubes. A similar reduction can be carried out for totally silver colourings: The *square* of a graph G , denoted by $G^{(2)}$, is the graph obtained from G by adding a new edge joining each pair of vertices at distance 2. The following is immediate from the definitions.

Proposition 2.3 *An r -regular graph G is totally silver if and only if $\chi(G^{(2)}) = r + 1$.*

This proposition shows a relation between our problem and the so-called hypercube colouring problems, namely colouring the graphs $Q_d^{(2)}$ and $Q_d^{(3)}$. These problems, introduced in the study of scalability of optical networks [11, 13], are also of interest in coding theory.

Any proper colouring of K_n is a silver $(n, 1)$ -cube. Silver $(n, 2)$ -cubes are silver matrices, which are studied in [7]. In this paper we study silver cubes of dimension at least 3.

The totally silver colouring problem is also related to domination in graphs. A set of vertices in a graph is a *dominating set* if every vertex outside the set has a neighbour in the set. The *domatic number* problem [4, §8.3] is that of partitioning the vertices of a graph into the maximum number of disjoint dominating sets. An easy upper bound on the domatic number of a graph G is $\delta(G) + 1$ where $\delta(G)$ is the minimum degree of G . A graph G with domatic number $\delta(G) + 1$ is said to be *domatically full*.

Proposition 2.4 *A regular graph has a totally silver colouring if and only if it is domatically full.*

Proof. If c is a totally silver colouring of G , then each colour class $c^{-1}(i)$ is a dominating set since every vertex with colour $j \neq i$, must be adjacent to a vertex with colour i by definition. Conversely, if $V(G) = S_0 \cup \dots \cup S_r$ is a partition of $V(G)$ into dominating sets where $r = \delta(G)$, then every vertex in S_i is adjacent to at least one vertex in each S_j with $j \neq i$. Therefore if we define $c(x) = i$ for all $x \in S_i$, all the colours $0, 1, \dots, r$ appear in the closed neighbourhood of every vertex. Thus c is a totally silver colouring of G . \square

3. A product construction

For any dimension d , the set of side lengths for which silver d -cubes exist is multiplicative.

Theorem 3.1 *If the (m, d) -cube and the (n, d) -cube are silver, then the (mn, d) -cube is silver.*

Before presenting the general proof, we illustrate the special case $m = 2, d = 3$. That is we use the silver $(2, 3)$ -cube of Figure 3.1 to double the side of any silver $(n, 3)$ -cube.

1	2	4	3
3	4	2	1

Fig. 3.1. The unique silver $(2, 3)$ -cube. A diagonal is indicated by shading.

Let $A : V(K_n^3) \rightarrow \{1, 2, \dots, 3n - 2\}$ be a silver colouring of K_n^3 with respect to a diagonal I . We aim to produce a silver colouring D of K_{2n}^3 . For $i = 0, 1, 2, 3$, let A_{i+1} be the colouring of K_n^3 obtained from A by adding ni to each colour k , where $2n - 1 \leq k \leq 3n - 2$. Let C_{i+1} be a proper colouring of K_n^3 with the colours $\{ni + 2n - 1, \dots, ni + 3n - 2\}$. Each of the eight blocks in Figure 3.2 represents an $n \times n \times n$ subcube of the cube of side $2n$. Let D be the colouring of K_{2n}^3 in which each subcube is coloured according to its label in Figure 3.2. We claim that D is a silver colouring.

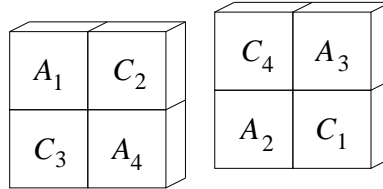


Fig. 3.2. Template for the doubling of a silver 3-cube.

Let J_i be the set of vertices in the subcube coloured by A_i which correspond to the diagonal I of A . We claim that D is silver with respect to the diagonal $J_1 \cup J_2 \cup J_3 \cup J_4$. Let $\mathbf{x} \in J_i$ for some $i = 1, 2, 3, 4$. Since A is silver, the colours $1, 2, \dots, 2n - 2$ and the colours $n(i - 1) + 2n - 1, \dots, n(i - 1) + 3n - 2$ all appear in the closed neighbourhood of \mathbf{x} in the subcube coloured by A_i . On the other hand the neighbourhood of \mathbf{x} in the subcube coloured by C_j is an n -clique for $j \neq i$ and is empty for $j = i$. Since C_j is a proper colouring of K_n^3 , for every $j \neq i$, the colours $n(j - 1) + 2n - 1, \dots, n(j - 1) + 3n - 2$ all appear in the neighbourhood of \mathbf{x} in C_j . Therefore all the colours $1, 2, \dots, 6n - 2$ appear in $N[\mathbf{x}]$.

The silver $(4, 3)$ -cube and $(6, 3)$ -cube presented in Figures 3.4 and 3.5 are obtained by applying this doubling construction to the silver 3-cubes of sides 2 and 3 presented in Figures 3.1 and 3.3.

Every diagonal I of K_n^3 is equivalent to a Latin square L of order n (specifically, $L_{ij} = k$ if and only if $(i, j, k) \in I$). For $n = 1, 2, 3$, all Latin squares of order n are equivalent (up to isomorphism of K_n^3). For $n = 4, 5$ there are exactly two such equivalence classes. This raises the question of the existence of silver cubes with prespecified diagonals. For $n = 4$, we

1	2	3	6	7	1	5	1	4
4	5	1	3	1	2	1	6	7
7	1	6	1	4	5	2	3	1

Fig. 3.3. A silver (3,3)-cube with respect to the (shaded) back-circulant diagonal.

①	2	5	6	4	③	6	5	9	10	①	2	10	9	8	⑦
3	④	6	5	②	1	5	6	10	9	7	⑧	9	10	②	1
7	8	①	2	8	7	10	⑨	1	②	3	4	⑥	5	4	3
8	7	9	⑩	7	8	②	1	⑤	6	4	3	2	①	3	4

Fig. 3.4. A (4,3)-cube which is silver with respect to two inequivalent diagonals. One is shaded, the other is circled.

1	2	3	8	9	10	6	7	1	9	10	8	5	1	4	10	8	9
4	5	1	9	10	8	3	1	2	10	8	9	1	6	7	8	9	10
7	1	6	10	8	9	1	4	5	8	9	10	2	3	1	9	10	8
11	12	13	1	2	3	12	13	11	15	16	1	13	11	12	14	1	4
12	13	11	4	14	1	13	11	12	3	1	2	11	12	13	1	15	16
13	11	12	16	1	15	11	12	13	1	4	14	12	13	11	2	3	1
14	15	16	1	2	3	15	16	14	12	13	1	16	14	15	11	1	4
15	16	14	4	11	1	16	14	15	3	1	2	14	15	16	1	12	13
16	14	15	13	1	12	14	15	16	1	4	11	15	16	14	2	3	1
1	2	3	5	6	7	9	10	1	6	7	5	8	1	4	7	5	6
4	8	1	6	7	5	3	1	2	7	5	6	1	9	10	5	6	7
10	1	9	7	5	6	1	4	8	5	6	7	2	3	1	6	7	5

Fig. 3.5. A silver (6,3)-cube obtained by doubling a silver (3,3)-cube.

see that the colouring of K_4^3 shown in Figure 3.4 is silver with respect to two inequivalent diagonals, one indicated by a gray shade and the other by circles. In Figures 3.6 and 3.7 we present two silver (5,3)-cubes with respect to the two inequivalent diagonals of K_5^3 . These silver cubes were found with the aid of Proposition 2.2 and by using graph colouring programs of [2].

A silver (7,3)-cube with respect to a back-circulant diagonal was found by Ventullo and Khodkar [10] during the revision of this paper. These examples and Theorem 3.1 imply the following.

Corollary 3.2 *The (n,3)-cube is silver for every positive side length n having no prime factor greater than 7.*

We do not know whether a silver (11,3)-cube exists.

Conjecture 3.3 *The (n,3)-cube is silver for every side length $n \geq 1$, and with respect to any diagonal of K_n^3 .*

The following proof of Theorem 3.1 is a straightforward generalization of the doubling construction discussed above. We partition the vertices of $G = K_{mn}^d$ into m^d blocks of size

1	2	3	4	5
6	1	10	11	7
7	10	1	13	2
8	7	13	12	9
9	11	6	8	13

10	3	2	7	4
12	9	5	10	8
8	6	4	3	12
1	2	11	4	13
2	5	13	6	11

11	5	13	1	6
9	11	12	2	3
13	3	7	9	8
2	12	4	3	10
7	4	1	12	9

12	4	8	10	1
10	5	6	13	12
1	2	9	5	3
6	8	5	11	4
4	3	7	1	2

13	6	11	8	9
2	7	8	1	4
11	1	10	6	13
5	10	2	13	7
3	12	4	9	5

Fig. 3.6. A silver $(5, 3)$ -cube with respect to the back-circulant diagonal I_0 .

1	2	3	4	5
6	11	4	9	3
7	10	12	5	6
8	1	2	13	10
9	8	13	12	7

10	11	4	7	12
2	13	1	8	6
11	3	9	12	1
5	8	7	3	13
8	6	3	5	11

11	9	8	2	10
5	12	7	4	13
6	11	1	8	9
10	4	5	7	2
2	13	12	3	1

12	5	13	9	3
10	7	9	11	8
13	12	11	10	5
1	3	8	6	4
4	11	10	1	2

13	1	6	8	2
3	5	13	10	11
4	2	8	13	7
2	6	12	11	9
7	9	2	6	4

Fig. 3.7. A silver $(5, 3)$ -cube with respect to a diagonal inequivalent to I_0 .

n^d indexed by the vertices of K_m^d and then colour the blocks according to the given silver cubes.

Proof of Theorem 3.1. We assume here that the vertex set of a complete graph K_t is $\{0, 1, \dots, t-1\}$. Let $G = K_{mn}^d$. For every $\mathbf{v} = v_1 v_2 \cdots v_d \in V(K_m^d)$ we let

$$T_{\mathbf{v}} = \{x_1 x_2 \cdots x_d \in V(G) \mid n v_i \leq x_i < n(v_i + 1) \text{ for } i = 1, 2, \dots, d\}.$$

Then $\{T_{\mathbf{v}} \mid \mathbf{v} \in V(K_m^d)\}$ is a partition of $V(G)$ and the induced subgraph $G[T_{\mathbf{v}}]$ is isomorphic to K_n^d for all \mathbf{v} . For every $\mathbf{v} \in V(K_m^d)$ we fix the isomorphism $\varphi_{\mathbf{v}} : T_{\mathbf{v}} \rightarrow V(K_n^d)$ given by $\varphi_{\mathbf{v}}(\mathbf{x}) = \mathbf{x} - n\mathbf{v}$ (here we regard \mathbf{x} and \mathbf{v} to be d -vectors of nonnegative integers). We define the projection $\varphi : V(G) \rightarrow V(K_n^d)$ by $\varphi(\mathbf{x}) = \varphi_{\mathbf{v}}(\mathbf{x})$ where \mathbf{v} is the unique vertex of K_m^d such that $\mathbf{x} \in T_{\mathbf{v}}$. We also define the *index*, $\text{idx}(\mathbf{x}) = \mathbf{v}$ where $\mathbf{x} \in T_{\mathbf{v}}$.

Let $A : V(K_n^d) \rightarrow \{1, 2, \dots, d(n-1) + 1\}$ and $B : V(K_m^d) \rightarrow \{0, 1, \dots, d(m-1)\}$ be silver colourings. Let I and J be respective diagonals of A and B . For every $\mathbf{v} \in V(K_m^d)$ we define two colourings $A_{\mathbf{v}}$ and $C_{\mathbf{v}}$ of K_n^d . For each \mathbf{v} , either $A_{\mathbf{v}}$ or $C_{\mathbf{v}}$ will be used to colour the block $T_{\mathbf{v}}$ of G . The colouring $C_{\mathbf{v}}$ is any proper colouring of K_n^d with colours

from the set $\{nB(\mathbf{v}) + (d-1)(n-1) + i \mid 1 \leq i \leq n\}$. For every $\mathbf{w} \in K_n^d$ we define

$$A_{\mathbf{v}}(\mathbf{w}) = \begin{cases} A(\mathbf{w}) & \text{if } A(\mathbf{w}) \leq (d-1)(n-1) \\ nB(\mathbf{v}) + A(\mathbf{w}) & \text{if } A(\mathbf{w}) > (d-1)(n-1). \end{cases}$$

We now define a colouring $D : V(G) \rightarrow \{1, 2, \dots, d(nm-1) + 1\}$. Given $\mathbf{x} \in V(G)$ with $\text{idx}(\mathbf{x}) = \mathbf{v}$, let

$$D(\mathbf{x}) = \begin{cases} A_{\mathbf{v}}(\varphi(\mathbf{x})) & \text{if } \mathbf{v} \in J \\ C_{\mathbf{v}}(\varphi(\mathbf{x})) & \text{if } \mathbf{v} \notin J. \end{cases} \quad (3.1)$$

We claim that D is a silver colouring of G with respect to the diagonal

$$K = \bigcup_{\mathbf{v} \in J} \varphi_{\mathbf{v}}^{-1}(I).$$

To prove the claim, let $\mathbf{x} \in K$, $\mathbf{v} = \text{idx}(\mathbf{x})$, and let $N[\mathbf{v}]$ be the closed neighbourhood of \mathbf{v} in K_m^d . Then $\mathbf{v} \in J$ and $\varphi_{\mathbf{v}}(\mathbf{x}) \in I$. For every $\mathbf{v}' \in N[\mathbf{v}]$ let $X_{\mathbf{v}'} = N_G[\mathbf{x}] \cap T_{\mathbf{v}'}$. By the definition we have

$$D(X_{\mathbf{v}'}) = \{nB(\mathbf{v}') + (d-1)(n-1) + i \mid 1 \leq i \leq n\}$$

for $\mathbf{v}' \neq \mathbf{v}$, and

$$D(X_{\mathbf{v}}) = \{1, 2, \dots, (d-1)(n-1)\} \cup \{nB(\mathbf{v}) + (d-1)(n-1) + i \mid 1 \leq i \leq n\}.$$

Now since $\mathbf{v} \in J$, we have $\{B(\mathbf{v}') \mid \mathbf{v}' \in N[\mathbf{v}]\} = \{0, 1, \dots, d(m-1)\}$, thus

$$D(N_G[\mathbf{x}]) = \bigcup_{\mathbf{v}' \in N[\mathbf{v}]} D(X_{\mathbf{v}'}) = \{1, 2, \dots, d(mn-1) + 1\}.$$

Therefore \mathbf{x} is rainbow with respect to D . □

Remark 3.4 The diagonal K defined in the above proof is the (generalized) Kronecker product of I and J . That is, $\mathbf{x} \in K$ if and only if, as vectors, $\mathbf{x} = \mathbf{w} + n\mathbf{v}$ for some $\mathbf{w} \in I$ and some $\mathbf{v} \in J$. Therefore K is not necessarily equivalent to a back-circulant even if both I and J are back-circulants.

Remark 3.5 The above proof remains valid if just before (3.1), for each $\mathbf{v} \in J$ we select an automorphism $\sigma_{\mathbf{v}}$ of K_n^d and modify both $A_{\mathbf{v}}$ and its diagonal through the action of $\sigma_{\mathbf{v}}$. In this way we can obtain colourings D which are silver with respect to inequivalent diagonals of K_{mn}^d .

4. Desarguesian silver cubes

In this section we consider the question: for which pairs (n, d) is K_n^d totally silver? We first give a necessary condition, then we show that this necessary condition is sufficient when n is a prime power by constructing a totally silver colouring. The construction uses perfect linear codes with suitable parameters. Finally, we use these totally silver colourings to construct some other silver cubes.

Proposition 4.1 *If G is r -regular, then in every totally silver colouring of G , each colour appears exactly $\frac{|V(G)|}{r+1}$ times. In particular, if a totally silver (n, d) -cube exists, then n^d is divisible by $d(n-1) + 1$.*

Proof. Let c be a totally silver colouring of G and let i be a colour. We count in two ways the pairs $(\mathbf{v}, \mathbf{v}')$ where $\mathbf{v} \in V(G)$, $\mathbf{v}' \in N[\mathbf{v}]$ and $c(\mathbf{v}') = i$. Each $\mathbf{v} \in V(G)$ contributes one to the count, whereas each vertex $\mathbf{v}' \in c^{-1}(i)$ contributes $r+1$ to the count. Therefore $|V(G)| = (r+1)|c^{-1}(i)|$. \square

Theorem 4.2 *Let q be a prime power. Then K_q^d is totally silver if and only if $d = \frac{q^s-1}{q-1}$ for some positive integer s .*

Proof. Let $q = p^t$ where p is a prime and t a positive integer. Suppose K_q^d has a totally silver colouring. By Proposition 4.1, $d(p^t-1) + 1$ divides p^{td} , so $d = \frac{p^k-1}{p^t-1}$ for some positive integer k , $t \leq k \leq td$. Since d is an integer we have, by an elementary argument, that t divides k . Therefore $d = \frac{q^s-1}{q-1}$ for some positive integer s .

For the converse, we note that for $d = \frac{q^s-1}{q-1}$, there exists a perfect linear code of length d with minimum distance 3, namely the $(d, d-s)$ Hamming code over \mathbb{F}_q [6, p. 191].

A totally silver colouring of K_q^d is defined by cosets of such a code. Since each coset is a code of distance at least 3, the closed neighbourhood of each vertex intersects each coset in at most (hence exactly) one vertex. Therefore this colouring is totally silver. \square

Theorems 3.1 and 4.2 give the following.

Corollary 4.3 *Let q be a prime power and let $s \geq 0$, $t \geq 1$ be integers. Then a silver $(q^s, \frac{q^t-1}{q-1})$ -cube exists.*

Starting with the construction of the proof of Theorem 4.2, we may produce silver cubes of ‘‘affine’’ dimension. These cubes are not totally silver.

Theorem 4.4 *Let q be a prime power and $s \geq 1$. Then there exists a silver (q, q^s) -cube with respect to the back-circulant diagonal I_0 , such that all vertices in I_0 receive the same colour.*

Proof. Let $F = \mathbb{F}_q$ be a field of order q . Let $d = \frac{q^s-1}{q-1}$. By Theorem 4.2, the graph $H = K_q^d$ has a totally silver colouring $c : V(H) \rightarrow \{1, 2, \dots, q^s\}$. Again we have identified $V(H)$ with the vector space F^d .

Let $G = K_q^{q^s}$ where $V(G) = F^{q^s}$. Since $q^s = (q-1)d + 1$, we may partition the coordinates of vectors $\mathbf{v} \in V(G)$ according to the pattern

$$\mathbf{v} = (v_1, v_2, \dots, v_{q^s}) = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{q-1}, b)$$

where $b \in F$ and each $\mathbf{x}_i \in V(H)$. We define two linear functions on $V(G)$, $\omega : V(G) \rightarrow F$ and $\mathbf{u} : V(G) \rightarrow V(H)$ by

$$\begin{aligned} \omega(\mathbf{v}) &= v_1 + v_2 + \dots + v_{q^s} \\ \mathbf{u}(\mathbf{v}) &= \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \dots + \alpha_{q-1} \mathbf{x}_{q-1}. \end{aligned}$$

Here $\{\alpha_1, \dots, \alpha_{q-1}\}$ is an arbitrary fixed enumeration of $F^* = F \setminus \{0\}$.

We aim to show that the function

$$A : V(G) \rightarrow \{\infty\} \cup (\{1, 2, \dots, q^s\} \times F^*)$$

$$A(\mathbf{v}) = \begin{cases} \infty & \text{if } \omega(\mathbf{v}) = 0 \\ [c(\mathbf{u}(\mathbf{v})), \omega(\mathbf{v})] & \text{if } \omega(\mathbf{v}) \neq 0 \end{cases}$$

is a silver colouring of G with respect to the back-circulant diagonal $I_0 = \{\mathbf{v} \in V(G) \mid \omega(\mathbf{v}) = 0\}$. Let $\mathbf{v} \in V(G) \setminus I_0$ and $\mathbf{v}' \in V(G)$ be two vertices of G which differ in either one or two coordinates. Suppose, for contradiction, that $A(\mathbf{v}) = A(\mathbf{v}')$. We immediately deduce $\mathbf{v}' \in V(G) \setminus I_0$, so

$$\omega(\mathbf{v}) = \omega(\mathbf{v}') \quad \text{and} \quad c(\mathbf{u}(\mathbf{v})) = c(\mathbf{u}(\mathbf{v}')).$$

Now $\mathbf{u}(\mathbf{v})$ and $\mathbf{u}(\mathbf{v}')$ are vertices of H which are at distance at most 2 in H . As c is a totally silver colouring of H , the second equation implies $\mathbf{u}(\mathbf{v}) = \mathbf{u}(\mathbf{v}')$. We define $\mathbf{w} = (w_1, w_2, \dots, w_{q^s}) = \mathbf{v} - \mathbf{v}'$. The vector \mathbf{w} has one or two nonzero entries, and satisfies

$$\omega(\mathbf{w}) = 0 \quad \text{and} \quad \mathbf{u}(\mathbf{w}) = (0, 0, \dots, 0).$$

The first equation implies that \mathbf{w} has exactly two nonzero entries, w_k, w_ℓ , which satisfy

$$w_k + w_\ell = 0. \tag{4.1}$$

The second equation implies that $1 \leq k, \ell \leq q^s - 1$, and that $k - \ell$ is an integer multiple of d . In particular, there exist field elements α_i, α_j such that

$$\alpha_i w_k + \alpha_j w_\ell = 0.$$

Comparing with (4.1), we conclude that $\alpha_i = \alpha_j$. Therefore $k = \ell$, a contradiction. We conclude that A is a silver colouring with respect to I_0 . \square

Applying Theorem 3.1 gives the following.

Corollary 4.5 *Let p be a prime and $s, t \geq 0$. Then a silver (p^s, p^t) -cube exists.*

5. Hypercubes and binary codes

Silver $(2, d)$ -cubes are of special interest since K_2^d is the hypercube Q_d . Here we find some nonexistence results, a relationship to general binary codes, and we propose some conjectures. It is proved in [11] (see also [5, Eq. (2)]) that $\chi(Q_d^{(2)}) = d + 1$ when $d + 1$ is a power of 2. So by Proposition 2.3, the hypercube Q_d is totally silver when $d + 1$ is a power of 2. Zhou [12, Cor. 2.6] extends this result by showing that every Cayley graph of degree $2^t - 1$ and having a transitive Abelian group of automorphisms is totally silver. From Theorems 4.2 and 4.4 we have the following.

Corollary 5.1 *For all $t \geq 0$, the hypercube Q_{2^t} is silver and the hypercube $Q_{2^{t+1}-1}$ is totally silver.*

Note that since Q_d is bipartite, it has only two diagonals, namely its partite sets. The set of all vertices with even (resp. odd) Hamming weight is called the even (resp. odd) diagonal and denoted by I_0 (resp. I_1).

In this section we present some nonexistence results for silver hypercubes.

Lemma 5.2 *If d is odd, then every silver colouring of Q_d is totally silver.*

Proof. Let $c : V(Q_d) \rightarrow \{0, 1, \dots, d\}$ be a silver colouring of Q_d with respect to the diagonal I , and let $\mathbf{w} \notin I$. We may assume $c(\mathbf{w}) = 0$. We aim to show \mathbf{w} is rainbow with respect to c .

Let G be the complete graph on the vertex set $\{1, 2, \dots, d\}$. We define the colourings $\psi_0 : V(G) \rightarrow \{0, 1, \dots, d\}$ and $\psi_1 : E(G) \rightarrow \{0, 1, \dots, d\}$ as follows. For every $i \in V(G)$, let \mathbf{w}_i be the vertex of Q_d obtained from \mathbf{w} by flipping the i th coordinate and define $\psi_0(i) = c(\mathbf{w}_i)$. Note for every $i \in V(G)$ we have $\mathbf{w}_i \in I$. For each edge $e = \{i, j\}$ of G let \mathbf{w}_{ij} be the vertex of Q_d obtained from \mathbf{w} by flipping the i th and the j th coordinates and define $\psi_1(e) = c(\mathbf{w}_{ij})$. It remains to show ψ_0 is a proper vertex colouring of G . We prove this in the following steps:

- For every i , we have $\psi_0(i) \neq 0$ since \mathbf{w}_i is adjacent to \mathbf{w} .
- For every $e = \{i, j\} \in E(G)$, \mathbf{w} and \mathbf{w}_{ij} are both adjacent to the rainbow vertex \mathbf{w}_i . Thus $\psi_1(e) = c(\mathbf{w}_{ij}) \neq c(\mathbf{w}) = 0$.
- Let $e = \{i, j\}$ and $e' = \{i, k\}$ be adjacent edges of G . Since \mathbf{w}_{ij} and \mathbf{w}_{ik} are both adjacent to the rainbow vertex \mathbf{w}_i , we have $\psi_1(e) = c(\mathbf{w}_{ij}) \neq c(\mathbf{w}_{ik}) = \psi_1(e')$. Therefore ψ_1 is a proper edge d -colouring of G .
- For every $i \in V(G)$ and $e = \{i, j\} \in E(G)$, since \mathbf{w}_i and \mathbf{w}_{ij} are adjacent in Q_d we have $\psi_0(i) \neq \psi_1(e)$. Now since ψ_1 is a proper edge colouring of G , $\psi_0(i)$ is the unique element of $\{1, 2, \dots, d\}$ which is not in $\{\psi_1(e) \mid e \text{ is adjacent to } i\}$.
- Since d is odd, every colour class $\psi_1^{-1}(k)$, $k \in \{1, 2, \dots, d\}$, is a matching of size $\frac{d-1}{2}$ in G . The unique vertex of G which is unsaturated in this matching must receive colour k in ψ_0 . Therefore ψ_0 is a proper vertex colouring of G . \square

We can now give a characterization of silver hypercubes of odd dimension.

Theorem 5.3 *If d is odd then Q_d is silver if and only if $d = 2^t - 1$ for some $t \geq 1$.*

Proof. By Corollary 5.1 if $d = 2^t - 1$ for some $t \geq 1$, then Q_d is totally silver. On the other hand if d is odd and Q_d is silver, then Q_d is totally silver by Lemma 5.2. Now by Proposition 4.1, $(d+1) \mid 2^d$, thus $d+1$ is a power of 2. \square

Silver hypercubes of even dimension are more difficult to characterize. Although we do not have a complete characterization, we can obtain nonexistence results using bounds from coding theory. Let $G_d = \text{SR}(Q_d, I_1)$, as defined in Section 2. Note that $V(G_d) = I_0$ is the set of words in $\{0, 1\}^d$ having even Hamming weight. Two vertices of G_d are adjacent if they have Hamming distance 2. Any two elements of an independent set I of G_d have Hamming distance at least 4. By suppressing a parity-check bit, we obtain from I a corresponding binary code of length $d-1$ and distance at least 3. This correspondence is bijective, therefore the maximum size of an independent set in G_d equals $A(d-1, 3)$. Here $A(d-1, 3)$ is a standard notation for the maximum size of a general binary code of length $d-1$ and distance 3.

By Proposition 2.2, a silver colouring of Q_d with respect to the diagonal I_1 corresponds to a partition of the vertices of G_d into $d+1$ independent sets, and this corresponds to a partition of $\{0, 1\}^{d-1}$ into binary codes of size at most $A(d-1, 3)$. We have proved the following.

Proposition 5.4 *If Q_d is silver then $A(d-1, 3) \geq \frac{2^{d-1}}{d+1}$.*

The following result is attributed to R. J. McEliece in [6, p. 545, Problem (22)].

Theorem 5.5 *If $d = 4k + 2$ for some $k \geq 1$, then $A(d - 1, 3) \leq \frac{2^{d-1}}{d+2}$.*

Corollary 5.6 *If $d = 4k + 2$ for some $k \geq 1$, then Q_d is not silver.*

In [9] it is shown that $A(11, 3) = 144$. Since $144 < \frac{2^{11}}{13}$, Proposition 5.4 gives the following.

Corollary 5.7 *The hypercube Q_{12} is not silver.*

In light of the above, we propose the following.

Conjecture 5.8 *The hypercube Q_d is silver if and only if $d = 2^t - 1$ or $d = 2^t$ for some $t \geq 0$.*

Conjecture 5.9 *If d is not a power of 2, then $A(d - 1, 3) \leq \frac{2^{d-1}}{d+1}$.*

Conjecture 5.9 is stronger than Conjecture 5.8 by way of Corollary 5.1, Theorem 5.3 and Proposition 5.4. Conjecture 5.9 is affirmed in case d is odd [1, Proposition 3.1], and in the case $d \equiv 2$ modulo 4, by Theorem 5.5. Since $A(11, 3) = 144$, this conjecture also holds for $d = 12$. Hamming codes over \mathbb{F}_2 are perfect, so $A(d - 1, 3) = \frac{2^{d-1}}{d}$ when d is a power of 2. Therefore it remains to settle Conjectures 5.9 and 5.8 for $d = 4k$ where $k \geq 5$ and k is not a power of 2.

6. Defining sets in graph colouring

Let c be a proper k -colouring of a graph G and let $S \subseteq V(G)$. If c is the only extension of $c|_S$ to a proper k -colouring of G , then S is a *defining set* of c . The minimum size of a defining set among all k -colourings of G is denoted by $\text{def}(G, k)$.

The study of $\text{def}(G, k)$ is nontrivial only for $\chi(G) \leq k \leq \Delta(G) + 1$, since $\text{def}(G, k)$ is not defined for $k < \chi(G)$, and since $\text{def}(G, k) = |V(G)|$ when $k > \Delta(G) + 1$. The case $k = \chi(G)$ is studied in [8]. A more general survey of defining sets in combinatorics appears in [3].

We show here a relationship between $\text{def}(G, \Delta(G) + 1)$ where $G = K_n^d$ and the existence of silver (n, d) -cubes.

Lemma 6.1 [7] *Let S be a defining set of an $(r + 1)$ -colouring c of an r -regular graph G . Then the complement of S is an independent set in G .*

Theorem 6.2 *A silver cube of side n and dimension d exists if and only if*

$$\text{def}(K_n^d, d(n - 1) + 1) = n^d - n^{d-1}.$$

Proof. The case $d = 2$ of this argument appears in [7]. Let c be a silver colouring with respect to a diagonal I of K_n^d . Then the set S of non-diagonal vertices is a defining set of c . Conversely, let c be a proper $(d(n - 1) + 1)$ -colouring of K_n^d which admits a defining set S of size $n^d - n^{d-1}$. By the above lemma, $V(K_n^d) \setminus S$ is an independent set of K_n^d . This independent set has cardinality n^{d-1} , so $V(K_n^d) \setminus S$ is a diagonal of K_n^d . For every $x \in V(K_n^d) \setminus S$, we have $|c(N(x))| = d(n - 1)$. Therefore every $x \notin S$ is rainbow with respect to c . \square

Theorem 6.2 and Lemma 6.1 serve to further justify our choice of the definition of “diagonal” that we use in this paper.

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