Balancing Covectors

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Abstract

Recently, Goddyn, Hochstättler and Hliněný proved that the oriented flow number introduced by Goddyn, Tarsi and Zhang for an oriented matroid of rank r is bounded by $14r^2 \ln r$. We improve this bound by showing that any oriented matroid without a coloop admits a reorientation such the imbalance of each covector is at most $3r - 1$. In particular this yields a new upper bound for the oriented flow number.

1 Introduction

This paper is concerned with a generalization of the graph chromatic number to geometric hyperplane arrangements and to oriented matroids. There are at least two ways to extend the definition of graph chromatic number to the class of oriented matroids [**?**, **?**]. One consequence of our main result is a significant improvement in the upper bound derived in [**?**] regarding the *oriented flow number*. Moreover, the argument in this paper is somewhat easier than the probablisitic method used in [**?**].

Although the main result uses the language of oriented matroids, the specialization of the result to *represented* oriented matroids is easily accessible to a nonspecialist.

Theorem 1. *Let A be a real matrix of rank $r \geq 2$ such that deleting any of its columns preserves its rank. It is possible to multiply some columns of A by -1 so that, for every vector $\mathbf{t}$ in the row space of the resulting matrix, the number of positive entries in $\mathbf{t}$ is at most $3r-2$ times the number of negative entries in $\mathbf{t}$.*

Before stating the general result, we review some terminology and notation for oriented matroids using a variation of that used in the standard reference [**?**]. A *signed subset* X of a ground set E is a subset $\underline{X} \subseteq E$ (the *support* of X) together with an ordered partition (X^+, X^-) of $\underline{X}$, where either of the parts may be empty. An oriented matroid $\mathcal{O}$ on the ground set E can be presented as a set $\mathcal{L} = \mathcal{L}(\mathcal{O})$ of *covectors*. Each covector in $\mathcal{L}$ is a signed subset of E, and $\mathcal{L}$ satisfies the covector axioms listed in [**?**, §4.1.1]. A *coloop* is a covector whose support has cardinality one. If $F \subseteq E$, then the *reorientation* $\mathcal{O}_F$ of $\mathcal{O}$ specified by F is the oriented matroid obtained from $\mathcal{O}$ by replacing each covector (X^+, X^-) by $(X^+ \triangle (F \cap \underline{X}), X^- \triangle (F \cap \underline{X}))$, where $\triangle$ is the symmetric difference operator. The *imbalance* of a signed subset X is defined to be

$$\mathrm{imbal}(X) = \min\left\{\frac{|\underline{X}|}{|X^+|}, \frac{|\underline{X}|}{|X^-|}\right\},$$

if both X^+ and X^- are non-empty. It is convenient to define the imbalance of X to be ∞ if exactly one of X^+, X^- is empty and 0 if both are empty.

Theorem 2. *Every oriented matroid $\mathcal{O}$ of rank $r \geq 2$ and having no coloop has a reorientation in which every covector has imbalance at most $3r-1$.*

In case $r = 1$, the best upper bound on imbalance is 3. Here $\mathcal{O}$ has only one covector, whose support has size, say, s. An "alternating reorientation" yields an imbalance of 2 if s is even, and $\frac{2s}{s-1} \leq 3$ if s is odd (see [**?**] for details).

We denote by $\mathcal{C}^*(\mathcal{O})$ the set of (signed) cocircuits of an oriented matroid $\mathcal{O}$. Since cocircuits are precisely the covectors of minimal non-empty support, as a corollary of Theorem 2 we derive a considerable improvement on the upper bound of the oriented flow number of an oriented matroid without coloop. For an oriented matroid $\mathcal{O}$ the *oriented flow number* $\phi_o(\mathcal{O})$ is defined as

$$\phi_o(\mathcal{O}) := \min_{F \subseteq E} \max_{D \in \mathcal{C}^*(\mathcal{O}_F)} \mathrm{imbal}(D).$$

This invariant is introduced in [?] as a generalization of (the dual of) the circular chromatic number. In [?] an upper bound of $14r^2 \ln r$ for an oriented matroid of rank r was derived using the probabilistic method. The method applied here is inspired by [?].

Corollary 3. *Every oriented matroid $\mathcal{O}$ of rank r and having no coloop satisfies $\phi_o(\mathcal{O}) \leq 3r - 1$.*

2 Proof of Theorem 2

Here are a few more facts and definitions. It is well-known that any oriented matroid without coloops admits a *totally cyclic* reorientation. In a totally cyclically oriented matroid, every covector X satisfies $X^+ \neq \emptyset \neq X^-$, and hence $\operatorname{imbal}(X) \leq |\underline{X}|$. A circuit C in a connected matroid M is *removable*, if $M \setminus C$ is still connected and the rank of $M \setminus C$ equals the rank of M. We denote by $c(M)$ the *circumference* of M, which is the cardinality of the largest circuit in M.

Lemma 4 (Lemos, Oxley[?])**.** *Let M be a connected matroid of rank r. If $|E(M)| \geq 3(r+1) - c(M)$, then M has a removable circuit.*

The *restriction* $\mathcal{O}|F$ of an oriented matroid $\mathcal{O}$ to a subset F of its ground set is the oriented matroid on ground set F whose covectors are $\{X|_F \mid X \in \mathcal{L}(\mathcal{O})\}$. Here $X|_F$ denotes the signed set with support $\underline{X} \cap F$ and partition $(X^+ \cap F, X^- \cap F)$. The following almost trivial observation will prove helpful.

Proposition 5. *Let X be a signed subset of E and $E = E_1 \dot\cup \ldots \dot\cup E_k$ a partition. Then* $\operatorname{imbal}(X) \leq \max\{\operatorname{imbal}(X|_{E_1}), \ldots, \operatorname{imbal}(X|_{E_k})\}$.

Proof of Theorem 2. Since no loop is contained in any covector, we may assume that M, the matroid underlying $\mathcal{O}$, has no loops. By Proposition 5 we may furthermore assume that M is connected.

If $|E| \leq 3r - 1$, then $|E| \geq 2$ since M has no coloops, and in any totally cyclic orientation of $\mathcal{O}$ we have $\operatorname{imbal}(X) \leq |\underline{X}| \leq |E| \leq 3r - 1$.

We assume $|E| \geq 3r$. Since $r \geq 2$, we have $c(M) \geq 3$ so

$$|E| \geq 3(r+1) - c(M),$$

and Lemma 4 yields a removable circuit. Inductively, we find circuits $C_1, \ldots, C_k$ which are pairwise disjoint such that, setting $L = E \setminus \bigcup_{i=1}^{k} C_i$, the restriction

$M|L$ is still connected of rank r and $|L| < 3r$. Next, we choose a reorientation $\tilde{\mathcal{O}}$ of $\mathcal{O}$ such that each of $\tilde{\mathcal{O}}|L, \tilde{\mathcal{O}}|C_1, \ldots, \tilde{\mathcal{O}}|C_k$ is totally cyclic.

Now, let $X \in \mathcal{L}(\tilde{\mathcal{O}})$ be an arbitrary covector. We partition X according to the above partition of the groundset $E = C_1 \dot\cup \ldots \dot\cup C_k \dot\cup L$ into parts

$$X_L = X|_L, \quad \text{and } X_i = X|_{C_i} \quad \text{for } 1 \leq i \leq k.$$

Each X_i is a covector in $\mathcal{O}|C_i$ and hence

$$\text{imbal}(X_i) \leq |\underline{X_i}| \leq |C_i| \leq r+1 \qquad \text{for } 1 \leq i \leq k$$

and similarly

$$\text{imbal}(X_L) \leq |\underline{X_L}| \leq |L| \leq 3r-1.$$

Finally, Proposition 5 yields

$$\text{imbal}(X) \leq \max\{\text{imbal}(X_L), \text{imbal}(X_1), \ldots, \text{imbal}(X_k)\} \leq 3r-1.$$

□

3 Concluding Remarks

Our proof suggests the definition of a seemingly new parameter related to the flow resp. chromatic number

$$\phi_{\mathcal{L}}(\mathcal{O}) := \min_{F \subseteq E} \max_{X \in \mathcal{L}(\mathcal{O}_F)} \text{imbal}(X)$$

which measures the possibility to balance all covectors in an oriented matroid. Clearly, $\phi_{\mathcal{L}} \geq \phi_o$. It is not difficult to compute that

$$\phi_{\mathcal{L}}(\mathcal{O}^*(K_n)) = \binom{n}{2} \approx r + \sqrt{2r}$$

whereas

$$\phi_o(\mathcal{O}^*(K_n)) = \chi(K_n) = n \approx \sqrt{2r}$$

where $r = \binom{n}{2} - n + 1$ is the rank of the cographic oriented matroid $\mathcal{O}^*(K_n)$ of the complete graph on n vertices. Hence, our bound for $\phi_{\mathcal{L}}(\mathcal{O})$ is within a linear factor of the best upper bound, wheras it may be the case that $\phi_o(\mathcal{O}) = O(\sqrt{r})$ (see [**?**]).