CIRCULAR COLOURING THE PLANE

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Abstract. The *unit distance graph* $\mathcal{R}$ is the graph with vertex set $\mathbb{R}^2$ in which two vertices (points in the plane) are adjacent if and only if they are at Euclidean distance 1. We prove that the circular chromatic number of $\mathcal{R}$ is at least 4, thus improving the known lower bound of 32/9 obtained from the fractional chromatic number of $\mathcal{R}$.

Key words. graph colouring, circular colouring, unit distance graph

AMS subject classifications. 05C15, 05C10, 05C62

1. Introduction. The *unit distance graph* $\mathcal{R}$ is defined to be the graph with vertex set $\mathbb{R}^2$ in which two vertices (points in the plane) are adjacent if and only if they are at Euclidean distance 1. Every subgraph of $\mathcal{R}$ is also said to be a *unit distance graph*. It is known that (cf. [1, 2])

$$4 \leqslant \chi(\mathcal{R}) \leqslant 7,$$

and that (cf. [3, pp. 59–65])

$$\frac{32}{9} \leqslant \chi_f(\mathcal{R}) \leqslant 4.36.$$

Here $\chi(\mathcal{R})$ denotes the chromatic number of $\mathcal{R}$, and $\chi_f(\mathcal{R})$ is the fractional chromatic number of $\mathcal{R}$ defined as follows: a *b–fold colouring* of a graph G is an assignment of sets of b colours to the vertices of G. The *fractional chromatic number* of G, denoted $\chi_f(G)$, is defined by

$$\chi_f(G) = \inf\{\frac{a}{b} \mid G \text{ has a } b\text{–fold colouring using } a \text{ colours}\}.$$

In this paper we study the circular chromatic number of the unit distance graph $\mathcal{R}$.

Let $r \geqslant 2$, $a, b \in [0, r)$, and $a \leqslant b$. We define the *circular distance* of a and b, denoted by $\delta(a, b) = \delta_r(a, b)$, to be $\min\{b-a, r+a-b\}$. One may identify the interval $[0, r)$ with a circle C^r with perimeter r and then $\delta(a, b)$ will be the distance between a and b in C^r. It is easy to see that δ satisfies the triangle inequality.

If $a, b \in [0, r)$ (or equivalently $a, b \in C^r$), we define the *circular interval from a to b*, denoted $[a, b]$, as follows (see Figure 1.1):

$$[a, b] = \begin{cases} \{x \mid a \leqslant x \leqslant b\} & \text{if } a \leqslant b, \\ \{x \mid 0 \leqslant x \leqslant b \text{ or } a \leqslant x < r\} & \text{if } a > b. \end{cases}$$

An *r-circular colouring* of a graph G, is a function $c : V(G) \to C^r$ such that for every edge xy in G, $\delta(c(x), c(y)) \geqslant 1$. The *circular chromatic number* of G, denoted by $\chi_c(G)$, is

$$\chi_c(G) = \inf\{r \mid G \text{ admits an } r\text{-circular colouring}\}.$$

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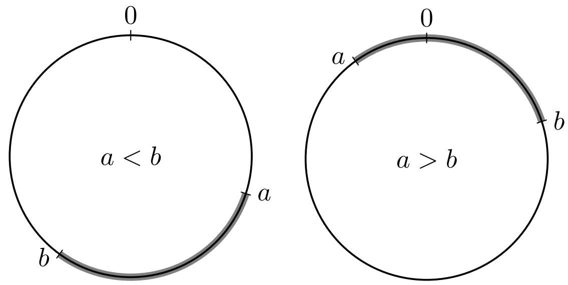

FIG. 1.1. *Circular intervals (clockwise direction is the positive direction)*

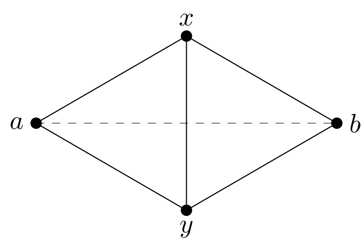

FIG. 2.1. *The unit distance graph $H_{a,b}$*

It is well known [4] that for every graph G, $\chi_f(G) \leqslant \chi_c(G) \leqslant \chi(G)$. For the unit distance graph $\mathcal{R}$, these inequalities give

$$\frac{32}{9} \leqslant \chi_f(\mathcal{R}) \leqslant \chi_c(\mathcal{R}) \leqslant \chi(\mathcal{R}) \leqslant 7.$$

We improve the lower bound for $\chi_c(\mathcal{R})$ to 4. We give two proofs of this result. The second one is constructive and gives a construction of finite unit distance graphs with circular chromatic number arbitrarily close to 4.

2. Proof. Let a and b be two points in the plane and let $d(a,b)$ denote the Euclidean distance between a and b. If $d(a,b) = \sqrt{3}$, then we may find points x and y in the plane such that the subgraph of $\mathcal{R}$ induced on the set $\{a,b,x,y\}$ is isomorphic to the graph H obtained by deleting one edge from K_4 (see Figure 2.1). We denote this unit distance graph by $H_{a,b}$. On the other hand, it is easy to see that in any embedding of H as a unit distance graph in the plane, the Euclidean distance between the two vertices of degree 2 in H is $\sqrt{3}$.

LEMMA 2.1. *Let $0 < \varepsilon < 1$ and $a, b \in \mathbb{R}^2$ with $d(a,b) = \sqrt{3}$. Let c be a $(3+\varepsilon)$-circular colouring of $H_{a,b}$. Then $\delta(c(a), c(b)) \leqslant \varepsilon$.*

Proof. Without loss of generality, we may assume $c(a) = 0$. Since a, x, y form a triangle in $H_{a,b}$, we have $c(x) \in [1, 1+\varepsilon]$ and $c(y) \in [2, 2+\varepsilon]$ up to symmetry. On the other hand, b is adjacent to both x and y. Thus

$$\begin{aligned} c(b) &\in [c(x)+1, c(x)-1] \cap [c(y)+1, c(y)-1] \\ &\subseteq [2, \varepsilon] \cap [-\varepsilon, 1+\varepsilon] \\ &= [-\varepsilon, \varepsilon]. \end{aligned}$$

The last equality is true since $1 + \varepsilon < 2$. ☐

THEOREM 2.2. $\chi_c(\mathcal{R}) \geqslant 4$.

Proof. Suppose that c is a $(3+\varepsilon)$-circular colouring of $\mathcal{R}$ where $0 \leqslant \varepsilon < 1$. Let

$$\mu = \sup\{\delta(c(a), c(b)) \mid a, b \in \mathbb{R}^2 \text{ and } d(a,b) = \sqrt{3}\}.$$

By Lemma 2.1, $\mu \leqslant \varepsilon$. By the definition of μ, for every $0 < \mu' < \mu$, there exist points a and b at distance $\sqrt{3}$ in the plane such that $\delta(c(a), c(b)) > \mu'$. Consider the graph $H_{a,b}$ as in Figure 2.1. Without loss of generality we may assume

$$0 = c(a) \leqslant c(b) < c(x) < c(y) \leqslant 2 + \varepsilon.$$

Since $3 + \varepsilon < 4$, we have

$$\delta(c(a), c(x)) = c(x) = \delta(c(a), c(b)) + \delta(c(b), c(x)) > \mu' + 1.$$

On the other hand since a and x are at distance 1, there exists a point z which is at distance $\sqrt{3}$ from both a and x. Therefore

$$1 + \mu' < \delta(c(a), c(x)) \leqslant \delta(c(a), c(z)) + \delta(c(z), c(x)) \leqslant 2\mu.$$

Since this is true for every $\mu' < \mu$, we have $\mu \geqslant 1$. This is a contradiction since $\mu \leqslant \varepsilon < 1$. ∎

3. A constructive proof. The graph $G_0 = K_2$ is obviously a unit distance graph. In our construction of graphs G_n $(n \geqslant 0)$ we distinguish two vertices in each of them. To emphasize the distinguished vertices x and y of G_n, we write $G_n^{x,y}$. We identify subgraphs of $\mathcal{R}$ with their geometric representation given by their vertex set.

For $n \geqslant 0$, the graph G_{n+1} is constructed recursively from four copies of G_n. Let $S = V(G_n^{x,y}) \subseteq \mathbb{R}^2$. Let us rotate the set S in the plane about the point x, so that the image y' of y under this rotation is at distance 1 from y. Let S' be the image of S under this rotation. Let T be the set of all points in $S \cup S'$ and their reflections across the line yy'. In particular let $z \in T$ be the reflection of x across the line yy'. We define $G_{n+1}^{x,z}$ to be the subgraph of $\mathcal{R}$ induced on T. This construction is depicted in Figure 3.1.

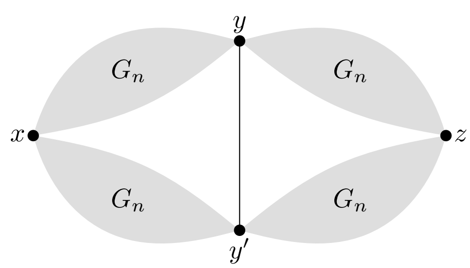

FIG. 3.1. *Construction of G_{n+1} from G_n*

Note that G_1 is the graph $H_{a,b}$ of Figure 2.1 and G_2 contains the Moser graph shown in Figure 3 as a subgraph. The Moser graph, also known as the spindle graph, was the first 4–chromatic unit distance graph discovered [2].

LEMMA 3.1. *For every $n \geqslant 1$, $\chi_c(G_n) \geqslant 4 - 2^{1-n}$. Moreover, for every $r = 4 - 2^{1-n} + \varepsilon$ with $0 \leqslant \varepsilon < 2^{1-n}$, and every circular r-colouring c of $G_n^{x,z}$, we have $\delta(c(x), c(z)) \leqslant 2^{n-1}\varepsilon$.*

Proof. We use induction on n. The nontrivial part of the case $n = 1$ is proved in Lemma 2.1. Let $n \geqslant 1$ and $G_{n+1}^{x,z}$ be as shown in Figure 3.1. Let $r = 4 - 2^{1-n} + \varepsilon$

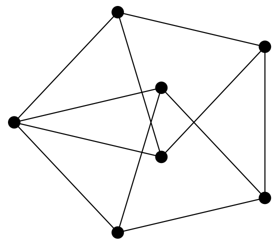

Fig. 3.2. *The Moser (spindle) graph*

for some $\varepsilon \geqslant 0$ and let c be a circular r-colouring of $G_{n+1}^{x,z}$. Without loss of generality we may assume that $c(x) = 0$. By the induction hypothesis, $\delta(0, c(y))$ and $\delta(0, c(y'))$ are both at most $2^{n-1}\varepsilon$. Hence $\delta(c(y), c(y')) \leqslant 2^n\varepsilon$. On the other hand, since y and y' are adjacent in $G_{n+1}^{x,z}$, we have $\delta(c(y)), c(y')) \geqslant 1$. Therefore $\varepsilon \geqslant 2^{-n}$ and we have $\chi_c(G_{n+1}) \geqslant 4 - 2^{1-n} + 2^{-n} = 4 - 2^{-n}$.

Now let $r = 4 - 2^{-n} + \varepsilon$ for some $0 \leqslant \varepsilon < 2^{-n}$, and let c be a circular r-colouring of G_{n+1} with $c(x) = 0$. Note that $r = 4 - 2^{1-n} + \varepsilon'$ with $\varepsilon' = 2^{-n} + \varepsilon < 2^{1-n}$. By the induction hypothesis, $\delta(0, c(y))$, $\delta(0, c(y'))$, $\delta(c(z), c(y))$ and $\delta(c(z), c(y'))$ are all at most $2^{n-1}\varepsilon' < 1$. Therefore we have

$$c(y), c(y') \in [-2^{n-1}\varepsilon', 2^{n-1}\varepsilon']$$

and

$$c(z) \in [c(y) - 2^{n-1}\varepsilon', c(y) + 2^{n-1}\varepsilon'] \cap [c(y') - 2^{n-1}\varepsilon', c(y') + 2^{n-1}\varepsilon'].$$

Since $\delta(c(y), c(y')) \geqslant 1$, one of $c(y)$ and $c(y')$, say $c(y)$, is in the circular interval $[-2^{n-1}\varepsilon', 2^{n-1}\varepsilon' - 1]$, and $c(y') \in [-2^{n-1}\varepsilon' + 1, 2^{n-1}\varepsilon']$. Therefore

$$[c(y) - 2^{n-1}\varepsilon', c(y) + 2^{n-1}\varepsilon'] \subseteq [-2^n\varepsilon', 2^n\varepsilon' - 1] = [-2^n\varepsilon', 2^n\varepsilon]$$

and

$$[c(y') - 2^{n-1}\varepsilon', c(y') + 2^{n-1}\varepsilon'] \subseteq [-2^n\varepsilon' + 1, 2^n\varepsilon'] = [-2^n\varepsilon, 2^n\varepsilon'].$$

Finally, since $\varepsilon' < 2^{1-n}$, we have $2^n\varepsilon' < r - 2^n\varepsilon'$. Hence

$$c(z) \in [-2^n\varepsilon', 2^n\varepsilon] \cap [-2^n\varepsilon, 2^n\varepsilon'] = [-2^n\varepsilon, 2^n\varepsilon].$$

This completes the induction step. □

Let us observe that, when constructing G_{n+1} from four copies of G_n, it may happen that vertices in distinct copies of G_n correspond to the same points in the plane. Additionally, it may happen that some edges between vertices in distinct copies of G_n are introduced. We may define in the same way a sequence of abstract graphs H_n, where none of these two issues occur. Clearly $\chi_c(G_n) \geqslant \chi_c(H_n)$, but we cannot argue equality in general. The proof of Lemma 3.1 applied to the graphs H_n gives slightly more:

Theorem 3.2. *For every $n \geqslant 0$, $\chi_c(H_n) = 4 - 2^{1-n}$.*

Proof. The cases $n = 0, 1$ are trivial. Let $n \geqslant 1$ and let H_{n+1} be as in Figure 3.1. Let $r = 4 - 2^{-n} = 4 - 2^{1-n} + 2^n$. By the proof of Lemma 3.1, $H_n^{x,y}$ admits a circular r-colouring c_1 with $c_1(x) = 0$ and $c_1(y) = \frac{1}{2}$. Similarly the graphs $H_n^{x,y'}$,

$H_n^{y,z}$ and $H_n^{y',z}$ admit circular r-colourings c_2, c_3 and c_4, respectively, with $c_2(x) = 0$, $c_2(y') = c_4(y') = -\frac{1}{2}$, $c_3(y) = \frac{1}{2}$, and $c_3(z) = c_4(z) = 0$. Now a circular r-colouring c of H_{n+1} can be obtained by combining the partial colourings c_1, c_2, c_3, c_4. □

The construction of this section gives an infinite subgraph of $\mathcal{R}$ with circular chromatic number at least 4. It remains open whether or not $\mathcal{R}$ has a finite subgraph with the same property.

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