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EULERIAN AND BIPARTITE ORIENTABLE MATROIDS

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Welsh [6] extended to the class of binary matroids a well-known theorem regarding Eulerian graphs.

Theorem 2.1 *Let M be a binary matroid. The ground set $E(M)$ can be partitioned into circuits if and only if every cocircuit of M has even cardinality.*

Further work of Brylawski and Heron (see [4, p. 315]) explores other characterizations of Eulerian binary matroids. They showed, independently, that a binary matroid M is Eulerian if and only if its dual, M^*, is a binary affine matroid. More recently, Shikare and Raghunathan [5] have shown that a binary matroid M is Eulerian if and only if the number of independent sets of M is odd.

This chapter is concerned with extending characterizations of Eulerian graphs via orientations. An Eulerian tour of a graph G induces an orientation with the property that every cocircuit (minimal edge cut) in G is traversed an equal number of times in each direction. In this sense, we can say that the orientation is *balanced*. Applying duality to planar graphs, these notions produce characterizations of bipartite graphs. Indeed the notions of flows and colourings of regular matroids can be formulated in terms of orientations, as was observed by Goddyn et al. [2]. The equivalent connection for graphs had been made by Minty [3].

In this chapter, we further extend these notions to *oriented matroids*. Informally, an oriented matroid is a matroid together with additional sign information. This is roughly analogous to orienting the edges in an undirected graph.

We assume that the reader is familiar with basic matroid theory. In Section 2.1, we develop a view of oriented matroids which is suited to our purposes, and which should be accessible to a reader familiar with graphs and matroids at the graduate level.

2.1 Orientations without vertices

The concept of orienting a graph can be understood by a child. Since a matroid has no vertices, one must work harder to understand oriented matroids. There are two equivalent definitions of oriented matroids: axiomatic and geometric. Each view offers advantages in understanding and working with these objects.

For completeness we give precise definitions of general oriented matroids, even though most of our results regard the simpler rank-3 case. The general definitions are a bit involved. However, the reader may elect to skim the second subsection below without risking a significant loss of understanding.

2.1.1 *Axiomatic definition*

An oriented matroid may be defined to be a matroid with additional sign information on its circuits and cocircuits. We use graph orientations to motivate and illustrate the definition. Recall that in a graphic matroid $M = M(G)$, the ground set is $E(G)$, the circuits of M are edge sets of graph cycles, and the cocircuits of M are the bonds (minimal edge cuts) of G. Throughout this chapter all graphs are assumed to be connected, but loops and multiple edges are allowed. Let $\mathcal{C} = \mathcal{C}(M)$ (and $\mathcal{B} = \mathcal{B}(M)$) denote the collections of circuits (and cocircuits) of a matroid M.

A *signing* of a set X is an unordered partition $\vec{X} = \{X^+, X^-\}$ of $X = X^+ \cup X^-$, where either part may be empty. A *signing* of a family $\mathcal{X}$ of sets is a set $\{\vec{X} \mid X \in \mathcal{X}\}$, where each $\vec{X}$ is a signing of X. Let G be a graph. Any orientation $\vec{G}$ of G naturally induces a signing $\vec{\mathcal{C}}$ of the family $\mathcal{C}$ of circuits of G, and induces a signing $\vec{\mathcal{B}}$ of the family $\mathcal{B}$ of cocircuits of G. In particular, each signed circuit $\vec{C} = \{C^+, C^-\}$ and signed cocircuit $\vec{B} = \{B^+, B^-\}$ in $\vec{\mathcal{C}} \cup \vec{\mathcal{B}}$ is determined by the directions in which edges of $\vec{G}$ traverse the cycle and the bond corresponding to C and B. The triple $(M(G), \vec{\mathcal{C}}, \vec{\mathcal{B}})$ is called the *(graphic) oriented matroid corresponding to* $\vec{G}$. This oriented matroid is denoted by $\vec{\mathcal{O}}(\vec{G})$. If $\vec{G}'$ is obtained from $\vec{G}$ by reversing all directed edges, then $\vec{\mathcal{O}}(\vec{G}) = \vec{\mathcal{O}}(\vec{G}')$. Conversely, one can prove that if two directed graphs $\vec{G}$ and $\vec{H}$ satisfy $\vec{\mathcal{O}}(\vec{G}) = \vec{\mathcal{O}}(\vec{H})$ and G is 2-edge connected, then $\vec{H} = \vec{G}$ or $\vec{H} = \vec{G}'$. Thus, the correspondence $\vec{G} \mapsto \vec{\mathcal{O}}(\vec{G})$ is 2-to-1 for 2-edge connected graphs G.

If a circuit C intersects a cocircuit B in $M(G)$, then the bond in G corresponding to B is crossed at least once in each direction when traversing C in G. We may restate this fact as follows. Let $\vec{G}$ be any orientation of G and let $(M, \vec{\mathcal{C}}, \vec{\mathcal{B}})$ be the oriented matroid corresponding to $\vec{G}$.

Then every pair $(\vec{C}, \vec{B}) \in \vec{\mathcal{C}} \times \vec{\mathcal{B}}$ satisfies the following.

$$(C^+ \cap B^+) \cup (C^- \cap B^-) = \emptyset \iff (C^+ \cap B^-) \cup (C^- \cap B^+) = \emptyset. \tag{2.1}$$

Any two signed sets $\vec{C}$, $\vec{B}$ which satisfy (2.1) are said to be *orthogonal*. It turns out that orthogonality is a characterizing property for those signings of graph circuits and cocircuits which are induced by graph orientations.

Proposition 2.2 *Let $\vec{\mathcal{C}}$ and $\vec{\mathcal{B}}$ be signings of the families of circuits and cocircuits of a graph G. Then $\vec{\mathcal{C}}$ and $\vec{\mathcal{B}}$ are both induced by some orientation of G if and only if every pair $(\vec{C}, \vec{B}) \in \vec{\mathcal{C}} \times \vec{\mathcal{B}}$ is orthogonal.*

This motivates the following definition, which is attributed to Bland and Las Vergnas [1, p. 118].

Definition 2.3 *An oriented matroid on the ground set E is a triple $\vec{\mathcal{O}} = (M, \vec{\mathcal{C}}, \vec{\mathcal{B}})$ where:*

1. *M is a matroid with ground set E, circuits $\mathcal{C}$ and cocircuits $\mathcal{B}$.*
2. *$\vec{\mathcal{C}} = \{\vec{C} \mid C \in \mathcal{C}\}$ and $\vec{\mathcal{B}} = \{\vec{B} \mid B \in \mathcal{B}\}$ are signings of the circuits and cocircuits of M such that each pair in $\vec{\mathcal{C}} \times \vec{\mathcal{B}}$ is orthogonal.*

The oriented matroid $\vec{\mathcal{O}} = (M, \vec{\mathcal{C}}, \vec{B})$ is called an orientation of M, and M is said to be orientable.

Not every matroid is orientable. For example the Fano matroid and its dual are not orientable. As we see shortly, every matroid M which is representable over the reals is orientable. Not every orientable matroid is representable over the reals.

To *reverse the orientation* of a set $F \subseteq E$ of elements of $\vec{\mathcal{O}}$ is to replace each signed circuit and cocircuit $\{X^+, X^-\} \in \mathcal{C} \cup \mathcal{B}$ with $\{X^+ \triangle F,\, X^- \triangle F\}$ where $\triangle$ denotes symmetric difference. For directed graphs, this operation corresponds to reversing the direction of the set F of edges in $\vec{G}$. For general oriented matroids, one can show that reversing the orientation of F preserves the orthogonality condition, and thus results in another oriented matroid. Any oriented matroid obtained from $\vec{\mathcal{O}}$ in this way is called a *reorientation* of $\vec{\mathcal{O}}$. Unlike directed graphs, reversing the orientation of *all* elements E of $\vec{\mathcal{O}}$ results in the same oriented matroid $\vec{\mathcal{O}}$. Thus reorienting on F yields the same oriented matroid as reorienting $E - F$. In fact, every connected oriented matroid $\vec{\mathcal{O}}$ of order n has exactly 2^{n-1} distinct reorientations. The set of reorientations of $\vec{\mathcal{O}}$ is called a *reorientation class* of M and is denoted by $\mathcal{O} = \mathcal{O}(M)$. This notation suggests that a graph G which underlies a directed graph $\vec{G}$ can be identified with the reorientation class $\mathcal{O}(\vec{G})$.

Unlike a graph, an orientable matroid M should not be identified with the reorientation class $\mathcal{O}$ of one of its orientations $\vec{\mathcal{O}}$. This is because M may have several orientations which belong to distinct reorientation classes. For example, the uniform matroid $U_{3,6}$ has precisely 4×2^5 distinct orientations, which are partitioned into 4 reorientation classes. Under the definition of *eulerian* that we propose below, only one of these four reorientation classes is eulerian. In other words, 'eulerian' is not a well-defined property of orientable matroids M. It is, however a well-defined property of a reorientation class $\mathcal{O}$.

2.1.2 *Geometric definition*

A topological description of oriented matroids was first given by Folkman and Lawrence, and independently by Edmonds and Mandel (again, see [1]). This definition is most accessibly introduced with reference to matroids represented by real matrices. Let A be an $r \times n$ real matrix of rank r. The matroid $M(A)$ *represented by* A has an element corresponding to each column of A. Independent sets in M correspond to linearly independent sets of columns of A. In fact, A determines an orientation $\vec{\mathcal{O}}(A)$ of $M(A)$ as follows. A circuit C is a minimally dependent set of columns. That is, a circuit is the support of a non-zero element of the *nullspace* $\{f \in \mathbb{R}^n \mid Af = 0\}$, where that support is minimal with respect to inclusion. If f, f' are two vectors in the nullspace supporting the same circuit C, then $f = \alpha f'$ for some non-zero scalar α. Thus the sign patterns f and f' are equal, up to total reversal of signs. In this way, a signing $\{C^+, C^-\}$ of each circuit C of $M(A)$ is well defined. Each cocircuit B of $M(A)$ is the support of an element of the *rowspace* $\{y^t A \mid y \in \mathbb{R}^r\}$, where that support is minimal with respect to inclusion. Again, the sign pattern of any rowspace vector with support B is unique up to total sign reversal. So the signing $\{B^+, B^-\}$ of B is well defined. Using the fact that the nullspace and rowspace of A are orthogonal vector subspaces, one easily verifies (2.1), so $\vec{\mathcal{O}}(A) = (M(A), \vec{\mathcal{C}}, \vec{\mathcal{B}})$ is an oriented matroid.

Reorienting $\vec{\mathcal{O}}(A)$ corresponds to multiplying some of the columns of A by -1. If each column of A contains at most one 1, at most one -1 and all other entries are 0, then $\vec{\mathcal{O}}(A)$ is the oriented matroid of a directed graph.

Suppose $\vec{\mathcal{O}} = \vec{\mathcal{O}}(A)$, where A is a real $r \times n$ matrix of rank r with no zero-column. Let $\mathbb{R}^r$ be the column space of A, and let $S = \{x \in \mathbb{R}^r \mid ||x|| = 1\}$ be the unit $(r-1)$-sphere. Let e be an element of $\vec{\mathcal{O}}$, so e is a column of A. Let S_e be the $(r-2)$-sphere consisting of points in S which are linearly orthogonal to e.

Each S_e is called a *hypersphere* of S. Each simply connected component of $S - S_e$ is homeomorphic to a $(r-1)$-ball, which is called a *side* of S_e. The side of S_e whose points have positive inner product with e is called the *positive* side of S_e, denoted by S_e^+; the other side, S_e^-, is the *negative* side of S_e. The intersection of any non-empty subset of $\{S_e \mid e \in E\}$ is a k-subsphere of S for some $0 \leq k \leq r-2$. The collection of all such subspheres is called the *sphere complex* represented by A, denoted by $\mathbb{S}(A)$. It is well known that the matroid $M(A)$ is faithfully encoded by the sphere complex $\mathbb{S}(A)$. For example, the rank of a set $F \subseteq E$ in $M(A)$ is precisely $r - 1 - k$, where the subsphere $\cap\{S_e \mid e \in F\}$ is a k-subsphere in $\mathbb{S}(A)$. Every 0-subsphere in $\mathbb{S}(A)$ is a pair of opposite points on S. The set of hyperspheres containing that 0-subsphere therefore corresponds to a maximal set F of matroid elements having rank $r-1$. That is, F is a *flat of rank* $(r-1)$ in the matroid $M(A)$. It is well known that cocircuits B of $M(A)$ are precisely sets of the form $E - F$ where F is a flat of rank $r-1$ in $M(A)$. Therefore cocircuits of $M(A)$ are easy to describe in terms of the sphere complex $\mathbb{S}(A)$: each 0-subsphere $S_0 \in \mathbb{S}(A)$ corresponds bijectively to the cocircuit $\{e \in E \mid S_0 \not\subseteq S_e\}$.

The points of $S \setminus \{S_e \mid e \in E(M)\}$ are partitioned into arcwise connected regions called *topes*. Each tope is homeomorphic to an $(r-1)$-ball.

Every point in the sphere can be encoded by a $\{+, -, 0\}$-valued vector as follows: first order the elements $S_1, \ldots, S_n$, of $\mathbb{S}$, then assign to entry i the value $+$ or $-$ whenever the point is in the positive or negative side of S_i, respectively, and 0 otherwise. These sign vectors are called *covectors*. Thus, the two points in a 0-subsphere have opposite covectors. Furthermore, the covector of a tope is well defined, since any two points in the same tope have equal covectors. The covector of a tope has no zero entries.

The orientation $\vec{\mathcal{O}}(A)$ of $M(A)$ is determined by the selection of positive and negative sides for each hypersphere S_e. Let $\vec{\mathbb{S}}(A) = (\mathbb{S}(A), f)$ where f maps each hypersphere S_e to its positive side S_e^+. The signing $\vec{B}$ of a cocircuit B is easy to recognize in $\vec{\mathbb{S}}(A)$. Let x be one of the two points which comprise the 0-sphere S_0 corresponding to $E - B$. We define $B^+ = \{e \in B \mid x \in S_e^+\}$, and $B^- = \{e \in B \mid x \in S_e^-\}$. Then $\{B^+, B^-\}$ is the signed cocircuit in the oriented matroid $\vec{\mathcal{O}}(A)$. Reorienting an element e in $\vec{\mathcal{O}}(A)$ corresponds to interchanging the positive and negative sides of S_e. The reorientation class $\mathcal{O}(A)$ containing $\vec{\mathcal{O}}(A)$ is thus faithfully represented by the hypersphere complex $\mathbb{S}(A)$.

The geometric description of a general (non-linear) oriented matroid of rank r is only slightly more involved than the sphere complex $\vec{\mathbb{S}}(A)$ of a matrix A. We still have a family $\{S_e \mid e \in E\}$ of hyperspheres in the unit $(r-1)$-sphere S, and the complex $\mathbb{S}$ of subspheres of S which are intersections of these hyperspheres. Each S_e is homeomorphic to an $(r-2)$-sphere. However, the hyperspheres S_e no longer need be linear; they may 'wobble' a bit. For this reason, each element of $\mathbb{S}$ is called a *pseudopsphere*. Each S_e is called a *pseudohypersphere*, and $\mathbb{S}$ is called a *pseudopsphere complex* or *PSC* for short. In order to avoid distracting topological complications, the unit sphere S is usually taken to be a piecewise linear set in $\mathbb{R}^r$ which is homeomorphic to an $(r-1)$-sphere. A formal definition follows.

Definition 2.4 *Let $\{S_e \mid e \in E\}$ be a finite family of pseudohyperspheres of a piecewise-linear $(r-1)$-sphere S such that*

1. *For every non-empty $F \subseteq E$, the intersection $S_F := \cap_{e \in F} S_e$ is homeomorphic to a k-sphere for some $0 \leq k \leq r-2$.*
2. *For every such non-empty intersection S_F and every $e \in E$ such that $S_F \not\subseteq S_e$, the intersection $S_F \cap S_e$ is a pseudosphere in S_F with sides $S_F^+ = S_F \cap S_e^+$ and $S_F^- = S_F \cap S_e^-$.*

The family $\mathbb{S} = \{S_F \mid \emptyset \neq F \subseteq E\}$ is called a pseudosphere complex (PSC) of rank r. The pair $\vec{\mathbb{S}} = (\mathbb{S}, f)$, where f maps each pseudohypersphere to its positive side is called an oriented PSC of rank r.

Theorem 2.5 (Edmonds et al.) *There is a bijective correspondence between PSCs $\mathbb{S}$ of rank r and reorientation classes $\mathcal{O}$ of rank r. Moreover, there is a bijective correspondence between oriented PSCs $\vec{\mathbb{S}}$ of rank r and oriented matroids $\vec{\mathcal{O}}$ of rank r.*

2.1.3 *Wiring diagrams*

This chapter is concerned primarily with oriented matroids of rank 3. A PSC of rank 3 is a family $\{S_e \mid e \in E(M)\}$ of simple closed curves in the 2-sphere S.

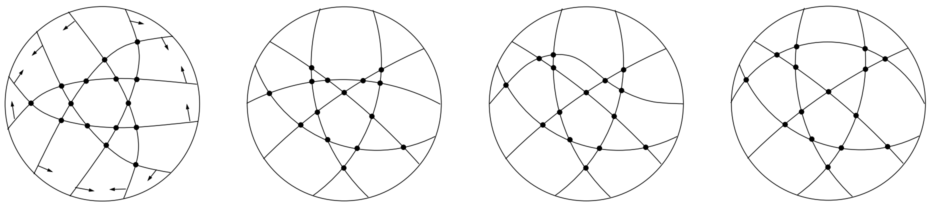

FIG. 2.1. Wiring diagrams for the four reorientation classes of $U_{3,6}$.

Each of these closed curves may be called a *pseudocircle*. Any two pseudocircles intersect in a 0-sphere. These 0-spheres naturally partition each pseudocircle into segments which we naturally view to be edges of a graph embedded on S whose vertex set is the union of the 0-spheres. The faces of this graph are the topes of the complex.

There is a convenient affine representation of a PSC of rank 3 which is called a *wiring diagram*. A wiring diagram is obtained from a rank-3 PSC as follows. The axioms ensure that there is another simple closed curve C in S which is in *general position*. That is, C intersects each pseudocircle S_e in two points (where they cross). Furthermore the two points comprising any 0-subsphere in the complex lie on opposite sides of C. By deleting one side of C, we obtain a disc D whose boundary is C. Within D is drawn a family of *pseudolines* or *wires* $\{W_e = S_e \cap D \mid e \in E(M)\}$. We usually draw D as a circular disc, and each pseudoline W_e is a curve joining opposite points of the boundary C. Any two pseudolines intersect at a point, and each such point x corresponds bijectively to the cocircuit $\{e \in E(M) \mid x \notin W_e\}$. An orientation of the PSC corresponds to a selection of a positive side W_e^+ for each pseudoline W_e. The signing $\{B^+, B^-\}$ of a cocircuit corresponding to x is determined by $B^+ = \{e \in E(M) \mid x \in W_e^+\}$. Because of the arbitrary choice in the selection of the C, different wiring diagrams may represent the same oriented matroid of rank 3. The wiring diagrams corresponding the four reorientation classes of the uniform matroid $U_{3,6}$ are depicted in Fig. 2.1.

2.1.4 *Eulerian and bipartite oriented matroids*

Definition 2.6 *Given an orientation $\vec{\mathcal{O}}$ of a reorientation class $\mathcal{O}$ of a matroid M, the discrepancy of a circuit is:* $\delta(C) = \left|\,|C^+| - |C^-|\,\right|$, *and that of a cocircuit B is:* $\delta(B) = \left|\,|B^+| - |B^-|\,\right|$.

Definition 2.7 *A reorientation class $\mathcal{O}$ is Eulerian if it admits an orientation $\vec{\mathcal{O}}$ where all cocircuits B satisfy $\delta(B) = 0$. It is bipartite if it admits an orientation $\vec{\mathcal{O}}$ such that all circuits C satisfy $\delta(C) = 0$.*

Graph theorists may recognize these definitions as extensions of the following elementary facts:

1. The graph G is Eulerian if and only if G has an orientation such that all cocircuits B, where B partitions the vertex set of G into $V = X \cup Y$, contain equal numbers of edges oriented *out* of X and *in* to X.
2. The graph G is bipartite if and only if it has an orientation such that, every circuit C contains equal numbers of 'clockwise' and 'anticlockwise' edges.

Recall that each cocircuit of $\mathcal{O}$ corresponds to a vertex (0-sphere) of the complex $\mathbb{S}$. Thus, a reorientation class $\mathcal{O}$ is Eulerian if it is possible to select, for each element in the PSC representing it, a positive side so that all the vertices in this configuration lie in equal numbers of positive and negative sides. That is, their covectors have equal number of + and − entries.

Regarding bipartite orientations, note that if a matroid contains a circuit C of size 3, then no orientation of the matroid makes $\delta(C) = 0$. This is a particular consequence of a parity condition that is born from our use of orientations to define bipartite and Eulerian matroids. It is a consequence of our use of bipartitions. Thus, to characterize simple, rank-3 bipartite matroids, we need only consider matroids of girth 4, which are orientations of the uniform matroid $U_{3,n}$. A 4-circuit C, in such a matroid is a set of 4 pseudolines and the partition $\{C^+, C^-\}$ is encoded in the arrangement as follows. In Fig. 2.2(a), the bold edges induce the signed circuit $(+, +, 0, -, -)$ (or $(-, -, 0, +, +)$) in $U_{3,5}$. The orientation is determined by comparison with the reference orientation indicated in Fig. 2.2(b). Once again, a graph theorist may be reminded of orientations of circuits, where edges are positively or negatively oriented depending on whether their orientation agrees, or disagrees with a reference orientation, namely clockwise or anticlockwise. We can fall back to the representable case to seek intuition of why the reference orientation is the one described in the figure. The positive sides of the hyperplanes in such a collection of planes determine the direction of the vectors representing the elements in C. Note that the positive half spaces of

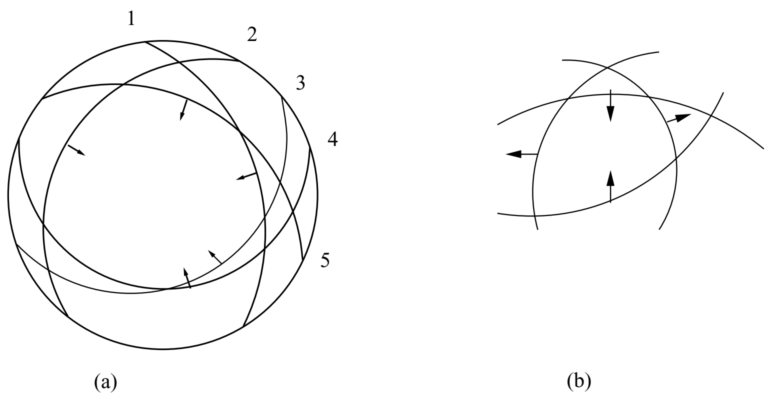

FIG. 2.2. (a) A 4-circuit in an oriented, uniform matroid. (b) The orientation of reference for such a 4-circuit.

these 4 planes cover the entire space. Hence, $\vec{0}$ is in the interior of the *positive cone* of these vectors, thus $\vec{0}$ is a positive linear combination of the vectors in C.

The circuit in Fig. 2.2 is *balanced,* that is, it has equal numbers of + and − signs. A cocircuit with this property is also called balanced.

2.2 Bipartite characterizations

Let $\mathcal{O}$ be a reorientation class, represented by a PSC. A tope of the PSC is *big* if every element of the matroid intersects the boundary of the tope in a facet (an $(r-2)$-cell). Thus a big tope in a rank-3, order-n matroid is a polygonal face of size n. Then we have the following:

Proposition 2.8 *A 4-circuit C in a rank-3, oriented matroid $\vec{\mathcal{O}}$ is balanced if and only if the restriction $\vec{\mathcal{O}}|_C$ has an all-positive big tope, that is, a 4-tope with covector* $(+,+,\cdots,+)$.

The restriction of $\vec{\mathcal{O}}$ to a set S is simply the matroid obtained by deleting all elements in the complement of S.

Proposition 2.9 *In a configuration with a big tope T, all edges on the border of T separate T from a triangular face.*

To see this consider the intersection of elements e_1, e_2 in Fig. 2.3. If the face adjacent to T, incident with e, in this configuration is not a triangle, then there must be at least one element f crossing e_1 and e_2, at some point between their intersection with e and with each other. Since f intersects e_1 and e_2 at these points, it cannot intersect them again, which prevents f from meeting T.

Theorem 2.10 *Let $\mathcal{O}$ be a reorientation class of a simple rank-3 matroid on n elements. $\mathcal{O}$ is bipartite if and only if the underlying matroid is uniform and the pseudosphere arrangement representing $\mathcal{O}$ has a big tope.*

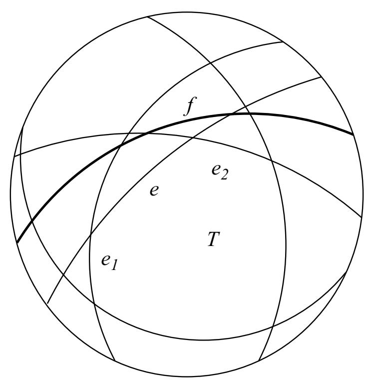

FIG. 2.3. A face adjacent to a big tope must be triangular.

Proof That we need only be concerned with uniform matroids follows from our comments at the end of the previous section.

Suppose that $\mathcal{O}$ has a big tope. Then orient all the elements so that this tope is all positive. Any 4 lines define a circuit, C, and the all-positive region in $\mathcal{O}|_C$ is 4-sided. By Proposition 2.8, these circuits are balanced, thus $\mathcal{O}$ is bipartite.

Conversely, suppose that $\mathcal{O}$ is bipartite. Take a bipartite orientation $\vec{\mathcal{O}}$ of $\mathcal{O}$, that is, one with the property that $|C^+| = |C^-|$ for all circuits C. Again, we know that the underlying matroid of $\mathcal{O}$ is uniform and we may assume that the number n of elements is at least 4. Otherwise, the matroid has no circuits. Further, if $n = 4$, the all-positive region is 4-sided, and there is nothing to show.

If $n > 4$, since $\mathcal{O}$ is bipartite, there exists a bipartite orientation $\vec{\mathcal{O}}$ and this orientation must be *acyclic* (has no totally oriented circuit). Thus it contains an all-positive tope, T [1, p. 122]. We must show that this tope is bounded by n lines.

Suppose, towards a contradiction, that T is bounded by fewer than n lines, which we label $e_1, \ldots, e_k$ cyclically around T. Thus, there is at least a line e not incident with T. In what follows, we are only concerned with the elements $e, e_1, \ldots, e_k$ and temporarily delete all other elements of $\mathcal{O}$. We will produce an unbalanced 4-circuit.

Since e does not intersect the tope T, it must intersect all elements bounding T at points outside the closure of T. If e goes through the (*triangular*) face adjacent to T, formed by 3 consecutive elements around T, then the 4-circuit indicated in darker lines in Fig. 2.4(a) is unbalanced in $\vec{\mathcal{O}}$ by Proposition 2.8. In fact, the same can be said if e goes through a triangular face formed by any 3 elements from $e, e_1, \ldots, e_k$. Otherwise, we have a situation that can be more clearly drawn with e as the outside circle (see Fig. 2.4(b)). Once again, the circuit

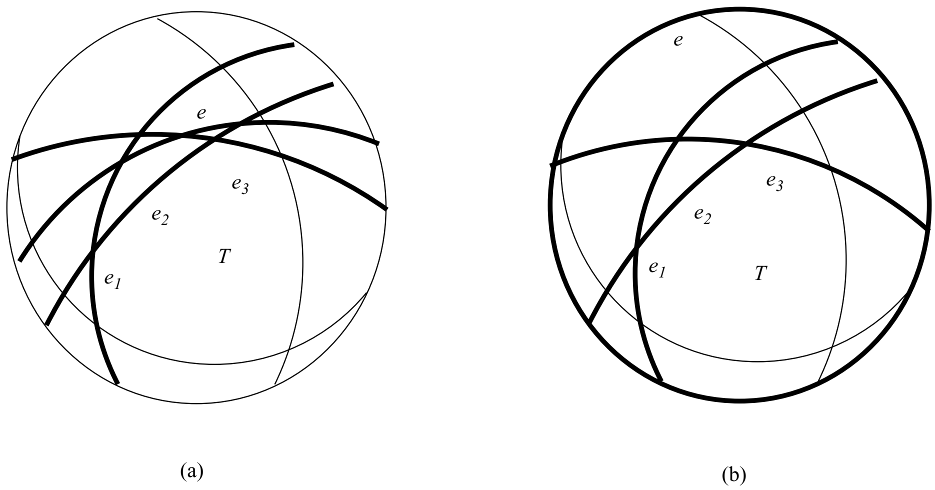

FIG. 2.4. Unbalanced 4-circuits in a matroid without a big tope.

pictured in darker lines is unbalanced, according to Proposition 2.8. Either way we obtain a contradiction. □

Unfortunately, Theorem 2.10 does not directly generalize to higher rank cases. Suppose that $\mathcal{O}$ is a representation of $U_{5,n}$ with a big tope. A circuit C of $\mathcal{O}$ is formed by any set of 6 elements and in the restriction of $\mathcal{O}$ to this circuit, any big tope is a 4-polytope which is isomorphic to the prism $T \times [0,1]$ over a tetrahedron T (see [7, p. 10]). The signing of such a circuit induces a partition $\{C^+, C^-\}$ no more balanced than one with sizes 4 and 2.

There exist, however, bipartite, higher rank, uniform, reorientation classes. For example, consider *alternating matroids,* denoted $C^{n,r}$, of odd rank r and order n (see [1, §8.2 and 9.4]). These are characterized by the fact the their element sets can be ordered in such a way that all bases are positively oriented. This, in turn, implies that the sign pattern on any circuit alternates. Since these matroids are uniform, circuits have size $r+1$, which is even. Thus, alternating matroids of odd rank are bipartite. If $r > 3$, $C^{n,r}$ is realizable and represented by a Vandermonde matrix,

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ t_1 & t_2 & \cdots & t_{n-1} & t_n \\ t_1^2 & t_2^2 & \cdots & t_{n-1}^2 & t_n^2 \\ \vdots & \vdots & & \vdots & \vdots \\ t_1^{r-1} & t_2^{r-1} & \cdots & t_{n-1}^{r-1} & t_n^{r-1} \end{pmatrix}$$

where $t_1 < t_2 < \cdots < t_n$. All submatrices are Vandermonde, hence have positive determinants.

2.3 Eulerian reorientation classes

Regarding characterizations of Eulerian, rank-3 matroids, we restrict our attention to uniform matroids. Consider the 2-dimensional cells T in the cell decomposition of a PSC representing an Eulerian, rank-3, oriented matroid $\mathcal{O}$. This orientation of $\mathcal{O}$ must be such that every T is bounded by a set of elements oriented in one of two possible ways, namely:

1. Alternating towards and away from T *(alternating).*
2. All pointing towards or all away from T *(consistent).*

To see this, recall that the orientation balances all vertices perfectly, so an exploration of adjacent vertices along a cell T shows that if two consecutive lines along the border of T are oriented consistently (both in or both out), then all other lines around T are also oriented consistently, while if they alternate, all other lines must also alternate. In Fig. 2.5 vertices u and v lie on the border of a (consistent) tope, T_1. Thus the only covector entries that differ between these two vertices correspond to the elements 2 and 3. If 1 and 2 are oriented consistently, 3 must also be consistent, so that the changing covector entries are 0,+ and +,0 as

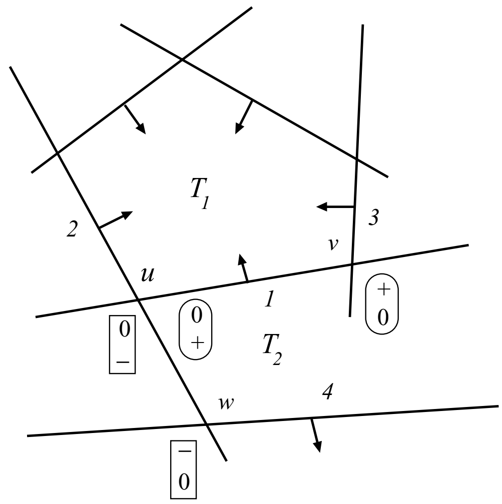

FIG. 2.5. A local exploration of the changes in sign vectors on adjacent vertices.

indicated. Now consider vertices u and w which are incident with a neighbouring tope, T_2. We must argue that T_2 is alternating. In this case, the covector entries that differ between u and w are those corresponding to elements 1 and 4. Thus the orientation of 1 and 2 force the orientation of 4 so that we get covector entries $0, -$ and $-, 0$ as indicated. Furthermore, note that two *adjacent* cells T_1, T_2 (separated by one element of $\mathcal{O}$) cannot be of the same type. Thus the 2-cells of such pseudosphere arrangements are properly 2-coloured, according to their type, as above. Indeed, if we regard this PSC as an embedded, planar graph, (vertices, the 0-dimensional cells, edges corresponding to 1-dimensional cells and faces, the 2-dimensional cells), we have a 4-regular graph which can be face coloured with two colours.

We can construct two graphs out of these arrangements, having vertices corresponding to the cells T with colour 1 and colour 2, respectively. Two vertices are adjacent if the corresponding cells are incident with a common vertex in the arrangement. One of these graphs, which we call G is bipartite and the other one, called H, is Eulerian. We will see that these graphs, in fact, provide a characterization of Eulerian reorientation classes of $U_{3,n}$.

As an example, note that a bipartite reorientation class of $U_{3,n}$, with n even, is also Eulerian. The Eulerian orientation is obtained by making the big tope alternating. One of the graphs associated to such a configuration is a *spiderweb* with n spokes and $n/2$ levels (see Fig. 2.6(a), and identify all the vertices in the outside face. Counting the high-degree vertices, we have a total of $n/2 + 1$ levels) and the other one is a cylindrical grid of width n and with $n/2$ levels (see Fig. 2.6(b)). These two graphs are simple, planar, and duals of each other.

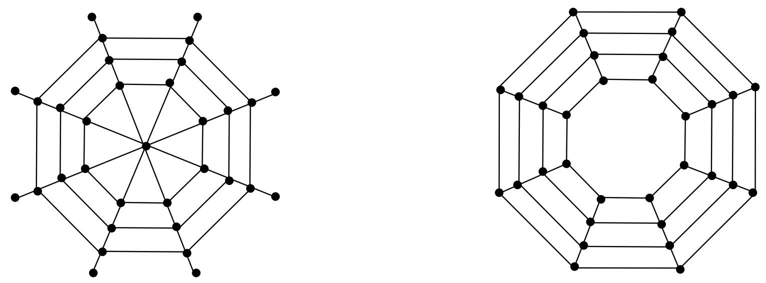

FIG. 2.6. The dual pair of plane graphs obtained from an Eulerian orientation of $U_{3,8}$.

Theorem 2.11 *A reorientation class $\mathcal{O}$ of the matroid $U_{3,n}$ is Eulerian if and only if n is even and the dual pair of graphs H and G, described above, form an Eulerian–bipartite pair.*

Proof First assume that $\mathcal{O}$ is an Eulerian reorientation class of $U_{3,n}$. We must show that n is even and that G and H are a bipartite–Eulerian pair.

All cocircuits of $U_{3,n}$ have cardinality $n-2$. Thus if $\mathcal{O}$ is Eulerian, it follows that n must be even. Since $\mathcal{O}$ admits an Eulerian orientation, let $\vec{\mathcal{O}}$ be such an orientation. From the discussion preceding this theorem, it follows that the topes of $\vec{\mathcal{O}}$ are all of the two types alternating and consistent. Furthermore, this partition defines the vertex set of a dual pair of graphs H and G. Since any alternating tope must be of even cardinality, the graph with vertex set corresponding to alternating topes is Eulerian. It follows that the other graph is bipartite.

Conversely, suppose that $\mathcal{O}$ is a representation of $U_{3,n}$ whose topes induce an Eulerian–bipartite pair of graphs. We must show that there is an orientation $\vec{\mathcal{O}}$ of $\mathcal{O}$, which balances all vertices of the configuration.

To describe this orientation, consider a two-colouring of the vertices of the bipartite graph and orient all elements incident with topes T, corresponding to vertices of colour one, so that T is in their positive side or *in*, and those of colour 2 so that T is in their negative side or *out*. This will induce an orientation for all elements in the matroid, provided we are able to show that it is well defined. That is, that once an element e is oriented one way, because of incidence with some tope T_1, there is no other tope T_2 forcing a different orientation on e. This follows from the fact that the graph defining the orientation is bipartite and such a contradiction would imply the existence of an odd circuit.

Finally, it must be shown that all vertices have zero discrepancy. It suffices to show that all vertices have equal discrepancy, since we know that antipodal vertices have discrepancies that are negatives of each other. This equality follows from the same analysis of adjacent vertices done before. These vertices appear consecutively around the border of some consistent tope and the covectors of these vertices differ in only two entries which are zero in one vector and both $+$, or both $-$, in the other (see Fig. 2.5). The observation that all vertices in the arrangement are incident with one such tope concludes the proof. □

Our goal now is to show that $U_{3,n}$ has a very large number of Eulerian orientations which belong to distinct orientation classes. As discussed is Section 2.1.3, reorientation classes for $U_{3,n}$ correspond to pseudocircle arrangements on a 2-sphere. These naturally correspond to certain 4-regular graphs embedded on the sphere, which is reminiscent of the theory of knots. By analogy with Reidemeister moves, there is a local transformation that changes one Eulerian orientation of $U_{3,n}$ into another Eulerian orientation of $U_{3,n}$ which belongs to a different reorientation class.

The transformation, which we call an *e-move*, involves four pseudocircles bounding a quadrilateral tope Q. An e-move at Q is defined if and only if Q is adjacent to two triangular topes, T_1, T_2. An e-move is depicted in Fig. 2.7(a). Since antipodal symmetry of a pseudosphere arrangement must be preserved, an e-move is simultaneously applied to the antipodal quadrilateral tope.

We consider the effect of an e-move on the associated plane bipartite graph G. This is illustrated in Fig. 2.7(b). Here G contains adjacent vertices t_1, t_2 of degree 3, which correspond to the triangular topes T_1, T_2. The edge $e = t_1t_2$ is incident to a face Q' of length 4 which corresponds to the quadrilateral tope Q.

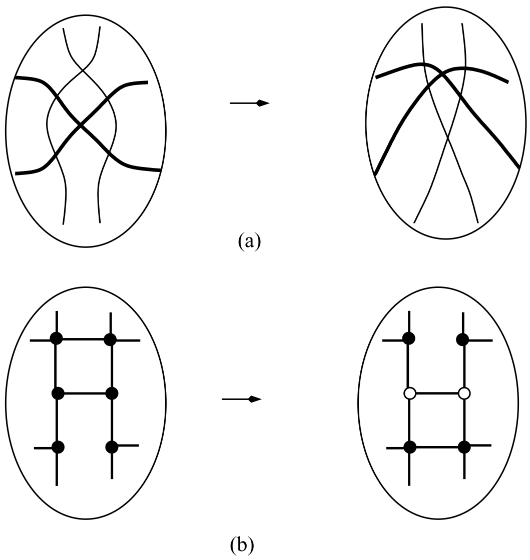

FIG. 2.7. An Eulerian orientation preserving transformation on the bipartite graph G.

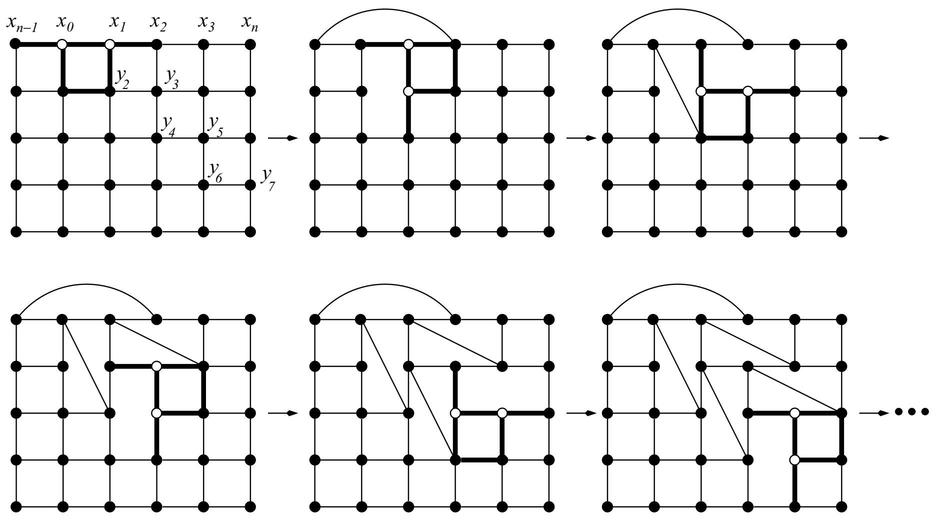

FIG. 2.8. A sequence of transformations in a subgraph of G.

Since two pseudocircles cross exactly twice, one easily sees that the other face of G incident with e has length at least 6. Let x, y be the neighbours of t_1, t_2 which lie on the boundary of the bigger face. The effect of performing an *e-move* at e is to delete e and add the edge xy. One might recognize that an e-move can be realized as a combination of four $\Delta - Y$ operations on the graph G (and, dually, on the graph H).

Let $n \geq 6$ be an even integer, and let $\vec{\mathcal{O}}$ be an Eulerian orientation of the reorientation class of $U_{3,n}$ with a big tope. The plane bipartite graph G associated with $\vec{\mathcal{O}}$ is depicted in Fig. 2.6(b) (for $n = 8$).

Let $x_0x_1x_2 \cdots x_{n-1}x_0$ be the cycle in G which bounds one of the two faces of length n. For some positive integer $k_0 \leq \frac{n-2}{2}$, let $y_2,\ y_3,\ \cdots,\ y_{k_0}$ a sequence of vertices in G as illustrated in Fig. 2.8. In that diagram we illustrate a sequence of e-moves, performed successively on the edges

$$x_0x_1, \quad x_1y_2, \quad y_2y_3, \quad y_3y_4, \ldots, \quad y_{k_0-1}y_{k_0}.$$

These moves are, of course, mirrored on the opposite side of the sphere. The final graph, G', in this sequence is the result of *growing a ladder of length* k_0 *starting from* x_0x_1.

We now select another integer $1 \leq k_1 \leq \frac{n-2}{2}$. Starting with G', we grow a ladder of length k_1 starting from x_3x_4. The two ladders we have grown do not interfere with each other, nor with the corresponding ladders on the other side of the sphere. Continuing in this way we see that, for any sequence of positive integers $k_0, k_1, \ldots, k_\ell$ where $\ell \leq \lfloor n/3 \rfloor$ and with each $k_i \leq \frac{n-2}{2}$, we may sequentially grow ladders of length k_i starting from $x_{3i}x_{3i+i}$. In Fig. 2.9 we present a schematic illustration of the result of this construction for some $n \geq 24$

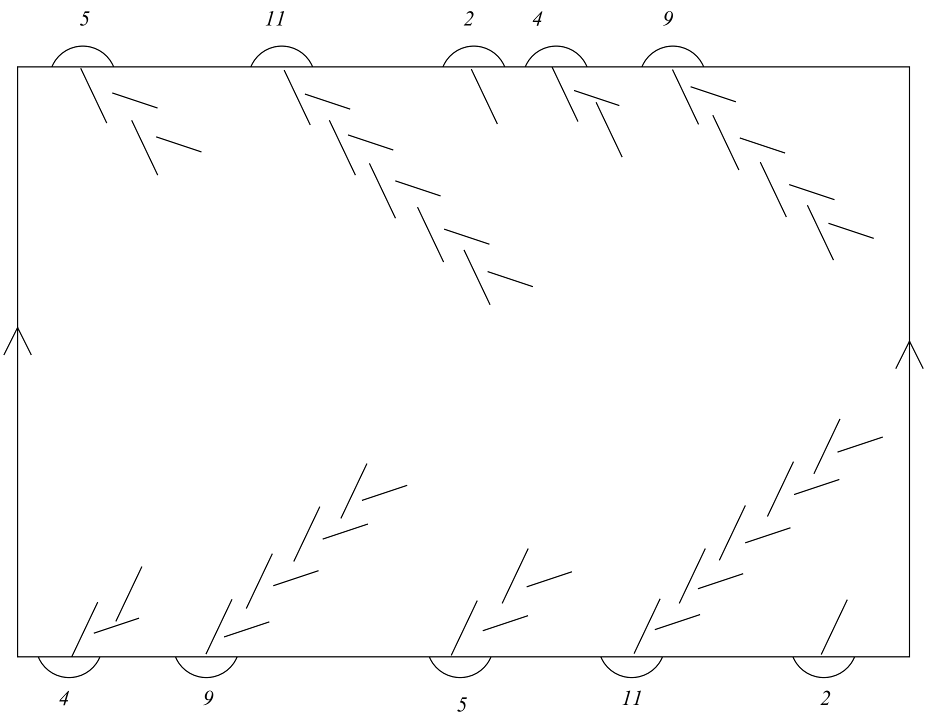

FIG. 2.9. A schematic look at the graph G associated with a reorientation class produced by a sequence of switches.

where $(k_0, k_1, k_2, k_3, k_4) = (5, 11, 2, 4, 9)$. Clearly, two sequences (k_i) and (k_i') will result in non-isomorphic graphs provided (k_i') is not a rotation or reflection of the sequence (k_i). Each such graph represents a distinct Eulerian reorientation class of $U_{3,n}$. Therefore there exist at least $\frac{1}{2}(\frac{n}{2})^{\lfloor n/3 \rfloor - 1}$ distinct Eulerian reorientation classes for $U_{3,n}$.

An interesting example is the matroid $U_{3,6}$. There are 4 reorientation classes of this matroid, only one of which is Eulerian. The graphs H and G associated with these classes are double covers of graphs in the Petersen family. That is, identifying antipodal vertices in G and H produces projective embeddings of these graphs. It is surprising, on the basis of small examples such as this one, that the number of Eulerian classes should grow superexponentially.

These are not the only Eulerian classes for $U_{3,n}$. Others can be derived from the configuration with the big tope using e-moves. For example, some ladders can be grown towards the South-West as well as towards the South-East. Still there is a construction due to D. Archdeacon, which results in a large family of Eulerian reorientation classes of $U_{3,n}$, none of which can be obtained by applying e-moves to the configuration with the big tope.

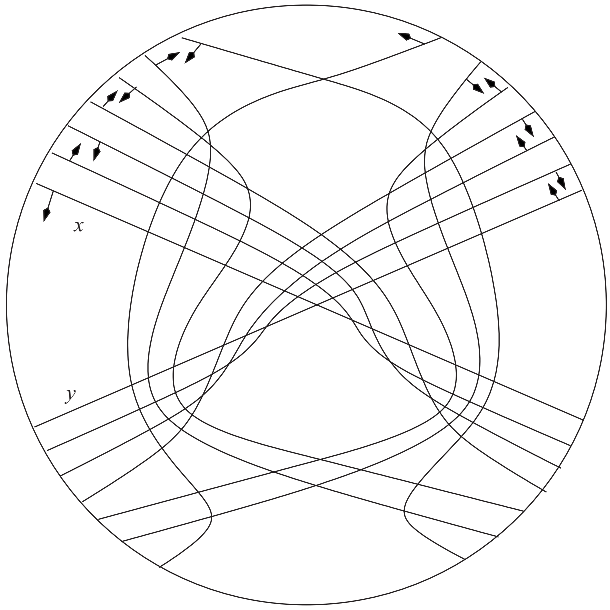

FIG. 2.10. The configuration associated with the sequence aaabba.

Consider a disc and take two diameters, x, y to be pseudolines. This cuts the perimeter of the disc in four intervals; North, South, East, and West. Now add, repeatedly, pairs of pseudolines which run from North to South, but are 'almost parallel' to x and to y. Each successive added pair are made to intersect either above (a) or below (b), all existing lines. In this way we construct one wiring diagram for each sequence of 'a' and 'b'. In Fig. 2.10 we see the diagram corresponding to the sequence aaabba. To see that each resulting configuration is Eulerian, we orient x and y so that the West interval is on the positive side of both pseudolines. The remaining pseudolines are alternatingly oriented starting from x and y as illustrated. It is not hard to see this is an Eulerian orientation of the wiring diagram. It is also not hard to see that this construction yields an exponential number of pairwise distinct reorientation classes for $U_{3,n}$, for even integers n.

2.4 Conclusions

Many questions remain unanswered in this chapter. Clearly there is much work to do on higher rank characterizations, and even in the non-uniform rank-3 case. Questions remain regarding uniform matroids of rank 3. For example, can one classify Eulerian orientations of $U_{3,n}$ up to equivalence under e-moves? This may be very difficult, for example, most configurations having the form of Fig. 2.10 appear to be pairwise non-equivalent under the transformation.

Evidently D. Welsh's early foray into extending graph properties to matroids has opened up a wealth of interesting and difficult questions.

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