Edge Disjoint Cycles Through Specified Vertices

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Abstract

We give a sufficient condition for a simple graph G to have k pairwise edge-disjoint cycles, each of which contains a prescribed set W of vertices. The condition is that the induced subgraph $G[W]$ be $2k$-connected, and that for any two vertices at distance two in $G[W]$, at least one of the two has degree at least $|V(G)|/2 + 2(k-1)$ in G. This is a common generalization of special cases previously obtained by Bollobás/Brightwell (where $k = 1$) and Li (where $W = V(G)$).

A key lemma is of independent interest. Let G be the complement of a bipartite graph with partite sets X, Y. If G is $2k$ connected, then G contains k Hamilton cycles which are pairwise edge-disjoint except for edges in $G[Y]$.

Keywords: Hamilton cycle, Hamilton circuit, connectivity, prescribed vertices, Ore condition, Fan condition, packing cycles, long cycle.

1 Introduction

In this paper we give a sufficient condition for a simple graph to have k pairwise edge-disjoint cycles, where every cycle contains a prespecified set of vertices. We state the main result.

Theorem 1 *Let $G = (V(G), E(G))$ be a finite undirected simple graph of order n, let $W \subseteq V(G)$, $|W| \geq 3$, and let k be a positive integer. Suppose that $G[W]$ is $2k$-connected, and that*

$$\max\{d_G(u), d_G(v)\} \geq n/2 + 2(k-1)$$

for every $u, v \in W$ such that $\mathrm{dist}_{G[W]}(u, v) = 2$. Then G contains k pairwise edge-disjoint cycles $C_1, \ldots, C_k$ such that $W \subseteq V(C_i)$, $1 \leq i \leq k$.

(Here, $d_G(v)$ is the degree of v in G, $G[W]$ is the subgraph induced by W, and $\mathrm{dist}_G(u, v)$ is the distance from u to v in G.)

The degree condition on W is in the spirit of Fan [2]. This *Fan-type* hypothesis gives a slightly stronger result than the corresponding *Ore-type* condition (that $d_G(u) + d_G(v) \geq n + 4(k-1)$

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for $u, v \in W$, $uv \notin E$). In [3], this degree condition was weakened (for sufficiently large n) to the best possible bound $d_G(u) + d_G(v) \geq n + 2(k-1)$.

Theorem 1 is a common generalization of previous results concerning the two special cases $k = 1$ and $W = V$. The case $k = 1$ is proved, in essence, by Bollobás and Brightwell [1]. Their result is stated with the Ore-type degree hypothesis, but does not require 2-connectivity. The special case $W = V$ is presented by Li [6] in 2000 as a slight sharpening of Li and Chen [7]. Some of our techniques are borrowed from [3, 6]. For further results on cycles in graphs we refer the reader to [4, 5].

The proof of Theorem 1 proceeds in two steps. First we prove the following lemma, which we regard to be of equal importance to the main theorem.

Lemma 2 *Let $G = (X \cup Y, E)$ be a $2k$-connected graph such that X and Y are disjoint cliques in G. Then G contains k Hamilton cycles $C_1, \ldots, C_k$ such that $e \in E(C_i) \cap E(C_j)$ implies $e \in E(G[Y])$, for $1 \leq i < j \leq k$. Moreover if $|Y| \geq k+1$, then each C_i contains an edge in $G[Y]$.*

The special case $|Y| = 1$ of Lemma 2 is (essentially) the well known decomposition of K_{2k+1} into Hamilton cycles. Our proof of Lemma 2 is inspired by Li's argument in [6]. However, Li requires the additional hypothesis $|Y| \geq 2k$. Dropping Li's hypothesis results in significant complications. The proof of Lemma 2 is presented in Section 3. The second step (Section 4) is to derive Theorem 1 from Lemma 2.

2 Notation and Auxiliary Results

All graphs $G = (V(G), E(G))$ are simple graphs. Let $x, y \in V(G)$ and let $X, Y \subseteq V(G)$. Then $\mathrm{dist}_G(x, y)$ is the distance from x to y in G. We denote by $N_G(X, Y)$ the set of vertices in Y which are adjacent in G to at least one vertex in X. We may write $N_G(x, Y)$ instead of $N_G(\{x\}, Y)$, and $N_G(x)$ instead of $N_G(x, V(G))$. We denote by $d_G(X, Y)$, $d_G(x, Y)$ and $d_G(x)$ the respective cardinalities $|N_G(X, Y)|$, $|N_G(x, Y)|$ and $|N_G(x)|$. The set of edges in G with one end in X and one end in Y is denoted $E_G(X, Y)$. We write $e_G(X, Y)$ for $|E_G(X, Y)|$. A u, v-path is a path whose endpoints are vertices u and v. A cycle $C \subseteq G$ goes *through* W if $W \subseteq V(C)$.

Let $Q \subseteq V(G)$. A vertex pair $\{a, b\} \subseteq V(G) - Q$ is *Q-linked in G* if there exist edges $e_1 = aq_1$, $e_2 = bq_2$ in G such that q_1, q_2 are distinct vertices in Q. A collection of subgraphs in G is *edge-disjoint outside of* Q if every edge of G which belongs to at least two of the subgraphs has both its endpoints in Q.

In Figure 2, we depict Walecki's famous decomposition of K_{2k+1} into Hamilton cycles, as described by Lucas [8].

Proposition 3 *For $\ell \geq 2k+1$, the graph K_ℓ contains k pairwise edge-disjoint Hamilton cycles.*

By deleting vertex $2k$ from Walecki's construction, we obtain a decomposition of K_{2k} into k Hamilton paths. Let $m \leq \lfloor k/2 \rfloor$, and consider the m Hamilton paths whose endpoints are the vertex pairs $\{1, 2\}$, $\{3, 4\}$, $\ldots$, $\{2m-1, 2m\}$. We observe that each of these Hamilton paths contains an edge joining 0 to a vertex in the set $\{k+1, k+3, k+5, \ldots, k+2m-1\}$. By relabeling vertices appropriately, the following result follows easily.

Proposition 4 *Let H be a complete graph of order n. Let $u_1, v_1, u_2, v_2, \ldots, u_k, v_k$ be distinct vertices of H, where $k \leq n/2$. Let $S \subseteq V(H)$ have cardinality $\geq m+1$, where $-1 \leq m \leq n/4$, and*

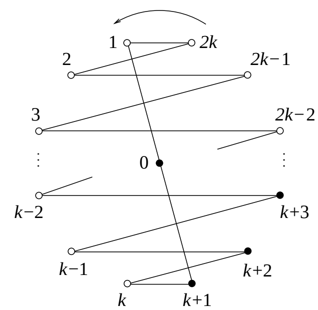


Figure 1: Walecki's decomposition: Rotate the depicted Hamilton cycle k times.

such that $u_j, v_j \notin S$ for $j \le m$. Then H has pairwise edge-disjoint Hamilton paths $P_1, \dots, P_k$, where P_i has endpoints u_i, v_i, $1 \le i \le k$, and where P_j contains an edge with both endpoints in S, $1 \le j \le m$.

Let Q, R, X be a partition of $V(G)$ so that X and $Q \cup R$ are both cliques in G. There are several places in this paper where we need to construct a Hamilton cycle in G starting with a Hamilton path P in $G - Q$. We give two constructions. Let $\{u, v\}$ be the endpoints of P.

Extension 1 Suppose $\{u, v\}$ is Q-linked in G. Then we may extend P to a Hamilton cycle of G by adding a Hamilton u, v-path in $G[Q \cup \{u, v\}]$.

Extension 2 Suppose u, v have a common neighbour $q \in Q$, and that there exists $e = ab \in E(P)$ where $a, b \in R$. Then we may extend $P - e$ to a Hamilton cycle of G by adding the path u, q, v and adding an a, b-Hamilton path in $G[Q - \{q\} \cup \{a, b\}]$. This construction makes sense even if $Q = \{q\}$.

Let $P_1, P_2, \dots, P_k$ be pairwise edge-disjoint Hamilton paths in $G - Q$ such that all $2k$ endpoints of these paths are distinct. If we can apply one of the above extensions to each path P_i, then the resulting cycles $C_1, \dots, C_k$ will be edge-disjoint outside of $Q \cup R$.

Lemma 5 *Let G be a graph with vertex partition $V(G) = X \cup Q \cup R$. Suppose that each of $X \cup R$ and $Q \cup R$ is a clique in G, and that $d_G(x, Q) \ge 1$ for $x \in X$. Suppose further that $X \cup R$ can be partitioned into k pairs of which at most $|R| - 1$ are not Q-linked in G. Then G contains k Hamilton cycles which are edge-disjoint outside of $Q \cup R$. Moreover, if $|Q \cup R| \ge 2$, then each of the Hamilton cycles contains an edge of $G[Q \cup R]$.*

Proof. If $|Q| = 1$, then $G = K_{2k+1}$. Moreover, assumptions of the lemma imply that $|R| \ge k+1$. Now we use Walecki's decomposition with $\{0, 1, \dots, k\} \subseteq R$ to obtain the required cycles.

We assume $|Q| \ge 2$. Suppose that $|R| \le \left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{|X \cup R|}{4} \right\rfloor$. We label the hypothesized pairs with $\{u_i, v_i\}$, $1 \le i \le k$, in such a way that $\{u_j, v_j\}$ is not Q-linked if and only if $1 \le j \le m$, for some $m \le |R| - 1$. Since $Q \cup R$ is a clique and $d_G(x, Q) \ge 1$ for $x \in X$, it follows that, for $1 \le j \le m$ we have $\{u_j, v_j\} \subseteq X$ and u_j, v_j have a common neighbour in Q. We apply Proposition 4 with $H = G[X \cup R]$ and $S = R$ to obtain edge-disjoint Hamilton paths $P_1, \dots, P_k$ in $G[X \cup R]$ where each P_i is a u_i, v_i-path and where each of $P_1, P_2, \dots, P_m$ has an edge in $G[R]$. Since $Q \cup R$ is a clique, we may apply Extension 2 to $P_1, \dots, P_m$ and apply Extension 1

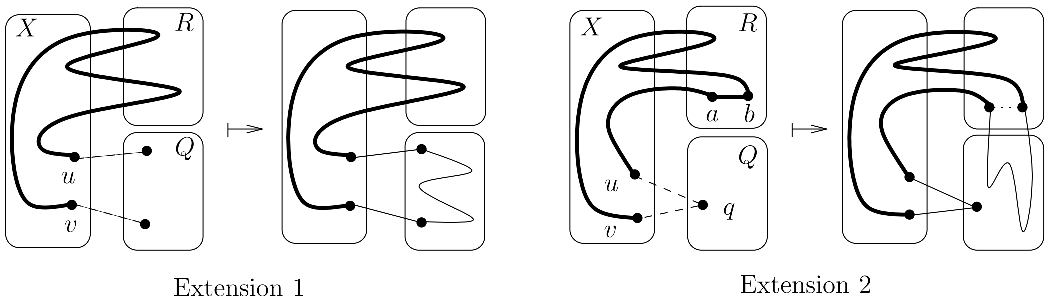


Figure 2: Using Extensions 1 and 2 to convert a u, v-path (shown in bold) into a Hamilton cycle of $G[X \cup Q \cup R]$. The vertices u and v may belong to either X or R.

to $P_{m+1}, \ldots, P_k$ to obtain k Hamilton cycles in G which are edge-disjoint outside of Q. See Figure 2. Each of these Hamilton cycles contains an edge of $G[Q \cup R]$, as required.

We now assume $|R| \geq \lceil \frac{k}{2} \rceil$. We partition $X \cup R$ into k pairs $\{u_i, v_i\}$, such that $u_i \in R$, $1 \leq i \leq k$. By Proposition 4 (with $S = \emptyset$), the subgraph $G[X \cup R]$ contains k pairwise edge-disjoint Hamilton paths P_i, $1 \leq i \leq k$, where each P_i is a u_i, v_i-path. Since $Q \cup R$ is a clique, $d_G(x, Q) \geq 1$ for $x \in X$, and since $|Q| \geq 2$, each pair $\{u_i, v_i\}$ is Q-linked. Now we may use Extension 1 to extend each P_i to a Hamilton cycle in G. The resulting Hamilton cycles are edge-disjoint outside of $Q \cup R$. Moreover, each cycle has an edge in $G[Q \cup R]$, as required. ■

3 Proof of Lemma 2

The basic idea used in the proof of Lemma 2 was introduced by Li [6] when he proved a weaker form of the lemma. Although our proof has details which are somewhat technical, the basic idea is not hard to describe. We first rearrange some edges of $G = (X \cup Y, E)$ using an operation called *edge flipping*. After performing a sequence of flips, we arrive at a new graph $G_{s,t}$ to which we may apply Lemma 5, finding k Hamilton cycles which are edge-disjoint outside of Y. Finally, the flipped edges are restored one by one, while modifying the Hamilton cycles appropriately. In the end, we obtain k Hamilton cycles in G which are edge-disjoint outside of Y, as desired.

Let $q, x, r \in V(G)$ be distinct vertices such that $qx \in E(G)$ and $xr \notin E(G)$. We define a new graph $G' = G - qx + xr$. We say that G' has been obtained from G by *flipping* the ordered triple $\langle q, x, r \rangle$. We denote this operation by $G \overset{qxr}{\longmapsto} G'$. Suppose now that $X \subseteq V(G) - \{q, r\}$. We may perform the series of flips $G \overset{qx_1r}{\longmapsto} G_1 \overset{qx_2r}{\longmapsto} \cdots \overset{qx_pr}{\longmapsto} G_p$ for any enumeration $x_1, x_2, \ldots, x_p$ of the set

$$X_{qr} = \{x \in X : qx \in E(G),\, xr \notin E(G)\}. \tag{1}$$

The resulting graph G_p is independent of the ordering $x_1, x_2, \ldots, x_p$. Therefore, the *multiflip* operation $G \overset{qXr}{\longmapsto} G_p$ is well defined for the ordered triple $\langle q, X, r \rangle$. We note that the result of a multiflip operation may leave the graph unchanged.

Let X, Q and R be disjoint subsets of $V(G)$. Let $\vec{Q} = (q_1, q_2, \ldots q_s)$ and $\vec{R} = (r_1, r_2, \ldots, r_t)$ be orderings (enumerations) of Q and R, respectively. The *$\vec{Q}X\vec{R}$-flip sequence* of G is the following sequence of multiflips, which is determined by the ordered triple $\langle \vec{Q}, X, \vec{R} \rangle$.

$$\begin{aligned} G &\overset{q_1 X r_1}{\longmapsto} G_{1,1} \overset{q_1 X r_2}{\longmapsto} G_{1,2} \overset{q_1 X r_3}{\longmapsto} \cdots \overset{q_1 X r_t}{\longmapsto} G_{1,t} \\ &\overset{q_2 X r_1}{\longmapsto} G_{2,1} \overset{q_2 X r_2}{\longmapsto} G_{2,2} \overset{q_2 X r_3}{\longmapsto} \cdots \overset{q_2 X r_t}{\longmapsto} G_{2,t} \\ &\vdots \\ &\overset{q_s X r_1}{\longmapsto} G_{s,1} \overset{q_s X r_2}{\longmapsto} G_{s,2} \overset{q_s X r_3}{\longmapsto} \cdots \overset{q_s X r_t}{\longmapsto} G_{s,t}. \end{aligned}$$

A graph $G_{i,j}$ in this sequence may be denoted by $G_{i,j}[\vec{Q} X \vec{R}]$ when the context is not clear.

Let $G = (X \cup Y, E)$ be a graph of order at least $2k+1$, where $G[X]$ and $G[Y]$ are disjoint cliques, and where $1 \leq |X| < 2k$. We select a subset R of Y so that $|X \cup R| = 2k$. We then select an ordering $\vec{R}$ of R and an ordering $\vec{Q}$ of $Q = Y - R$. Let $s = |Q|$, and let $t = |R|$. It is possible to make these selections in such a way that the graph $G_{s,t}[\vec{Q} X \vec{R}]$ has a special linking property. A variation of the following result (where the connectivity condition is replaced by strong degree conditions) appears as Proposition 2 of [6].

Lemma 6 *Let $G = (X \cup Y, E)$ be a $2k$-connected graph of order at least $2k+2$, where $G[X]$ and $G[Y]$ are disjoint cliques, and where $1 \leq |X| < 2k$. Then there exist a subset $R \subseteq Y$ having size $t = 2k - |X|$, an ordering $\vec{R}$ of R, and an ordering $\vec{Q}$ of $Q = Y - R$ such that the set $X \cup R$ can be partitioned into k pairs of which at most $t-1$ are not Q-linked in $G_{s,t}[\vec{Q} X \vec{R}]$, where $s = |Q|$.*

Proof. We first prove the lemma for $k = 1$. Suppose that $X = \{x\}$. By the 2-connectivity of G, there exist two vertices $a, b \in N_G(x, Y)$. We select $R = \{a\}$ and an arbitrary ordering $\vec{Q}$ of Q. We have $G_{s,t}[\vec{Q} X \vec{R}] = G$. Since $|Q| \geq 2$, and since $G[Y]$ is a clique, there is a vertex in $Q - \{b\}$ which is adjacent to a. Therefore $X \cup R = \{x, a\}$ is a Q-linked pair in $G_{s,t}$.

We assume $k \geq 2$. Suppose by way of contradiction that the lemma is false. Let k be the smallest integer such that there exists a $2k$-connected counterexample G. We shall further suppose that G has as many edges as possible.

Claim 1 No vertex $x \in X$ satisfies $2k+1 \leq d_G(x) \leq |V(G)| - 2$.

Suppose by way of contradiction that $x \in X$ satisfies $2k+1 \leq d_G(x) \leq |V(G)| - 2$. Then x is not adjacent to some $y \in Y$. The graph $G' = G + xy$ together with the sets X and Y satisfy the hypothesis of the lemma. By the maximality of $|E(G)|$, there exist ordered sets $\vec{R}$, $\vec{Q}$, defining a $\vec{Q} X \vec{R}$-flip sequence on G' such that $X \cup R$ can be partitioned into k pairs $\{u_i, v_i\}$ $(i = 1, 2, \ldots, k)$ of which at most $t-1$ are not Q-linked in $G'_{s,t}[\vec{Q} X \vec{R}]$. Consider now the graph $G_{s,t} = G_{s,t}[\vec{Q} X \vec{R}]$ and the same partition $\{u_i, v_i\}$ $(i = 1, 2, \ldots, k)$. Evidently $G_{s,t} = G'_{s,t} - xy'$ for some vertex $y' \in Q \cup R$. Without loss of generality, we suppose that $x = u_1$. For $i = 2, 3, \ldots, k$, the pair $\{u_i, v_i\}$ is Q-linked in $G_{s,t}$ if and only if it is Q-linked in $G'_{s,t}$. If $\{u_1, v_1\}$ is not Q-linked in $G'_{s,t}$, then we have proved the claim. Therefore we assume that $\{u_1, v_1\}$ is Q-linked in $G'_{s,t}$. We show that $\{u_1, v_1\}$ is also Q-linked in $G_{s,t}$. Since $d_{G'_{s,t}}(u_1) = d_{G'}(u_1) \geq 2k+2$, it follows that $d_{G'_{s,t}}(u_1, Q) \geq 3$. Therefore $d_{G_{s,t}}(u_1, Q) \geq 2$, so $\{u_1, v_1\}$ is a Q-linked pair in $G_{s,t}$. Therefore G is not a counterexample, proving Claim 1.

Claim 2 Every vertex $x \in X$ satisfies $d_G(x) = 2k$.

Suppose by way of contradiction that $d_G(x) \neq 2k$ for some $x \in X$. By Claim 1 and since G is $2k$-connected, we have $d_G(x) = |V(G)| - 1$. Suppose $1 \leq |X| \leq 2$. Then we define $G_{s,t} = G_{s,t}[\vec{Q}X\vec{R}]$, where $\vec{Q}$, $\vec{R}$ are selected arbitrarily subject to $Q \cup R = Y$, $Q \cap R = \emptyset$, and $|R| = t$. Since $|Q| \geq 2$, and $Q \subseteq N_{G_{s,t}}(x, Q)$, and $d_{G_{s,t}}(x', Q) \geq 1$ for $x' \in X - \{x\}$, any partition of $X \cup R$ into k pairs constitutes k Q-linked pairs in $G_{s,t}$, a contradiction.

We assume $|X| \geq 3$. Let $x' \in X - \{x\}$. Consider the graph $G' = G - \{x, x'\}$ and the partition (X', Y) of $V(G')$, where $X' = X - \{x, x'\}$. Then G' is a $2(k-1)$-connected graph of order at least $2(k-1)+2$, in which $G'[X']$ and $G'[Y]$ are cliques, and where $1 \leq |X'| < 2(k-1)$. By choice of G, there exist ordered sets $\vec{Q}$, $\vec{R}$ and a partition of $X' \cup R$ into $k-1$ pairs $\{u_i, v_i\}$ $(1 \leq i \leq k-1)$ of which at most $t'-1$ are not Q-linked in $G'_{s,t'}[\vec{Q}X'\vec{R}]$. (Here we have $t' = 2(k-1) - |X'| = t$.) Consider the graph $G_{s,t} = G_{s,t}[\vec{Q}X\vec{R}]$, and the partition of $X \cup R$ given by $\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_{k-1}, v_{k-1}\}, \{x, x'\}$. Obviously, for $i = 1, 2, \ldots, k-1$, the pair $\{u_i, v_i\}$ is Q-linked in $G_{s,t}$ if and only if it is Q-linked in $G'_{s,t}$. We have that $d_{G_{s,t}}(x') = d_G(x') \geq 2k > |X \cup R - \{x'\}|$, so $d_{G_{s,t}}(x', Q) \geq 1$. Since $Q \subseteq N_{G_{s,t}}(x)$ and $|Q| \geq 2$, the pair $\{x, x'\}$ is Q-linked in $G_{s,t}$, a contradiction. This proves Claim 2.

Let us label the vertices in Y with $r_1, r_2, \ldots, r_t, q_1, q_2, \ldots, q_s$ in such a way that

$$d_G(r_1, X) \geq d_G(r_2, X) \geq \cdots \geq d_G(r_t, X) \geq d_G(q_1, X) \geq d_G(q_2, X) \geq \cdots \geq d_G(q_s, X).$$

Let $\vec{R} = r_1, r_2, \ldots, r_t$ and $\vec{Q} = q_1, q_2, \ldots, q_s$ be orderings of the sets $R = \{r_1, r_2, \ldots, r_t\}$ and $Q = Y - R$. We aim to show that the graph $G_{s,t} = G_{s,t}[\vec{Q}X\vec{R}]$ satisfies the conclusion of Lemma 6 for some partition of $X \cup R$ into pairs.

Since $|X \cup R| = 2k$, it follows from Claim 2 and the nature of the flipping procedure that $X \cup R$ is a clique in $G_{s,t}$ and that

$$d_{G_{s,t}}(x, Q) = 1 \text{ for all } x \in X. \tag{2}$$

Let $S = \{\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_k, v_k\}\}$ be a partition of $X \cup R$ into k pairs, such that the number, m, of pairs in S which are not Q-linked in $G_{s,t}$ is minimized. We assume that $\{u_j, v_j\}$ is not Q-linked if and only if $1 \leq j \leq m$. Since G is a counterexample, we have $1 \leq t \leq m$, so $\{u_1, v_1\}$ is not Q-linked. Since $Q \cup R$ is a clique, $|Q| \geq 2$, and by (2), we have $u_1, v_1 \in X$ and u_1, v_1 have a common neighbour, say q_{i_0}, in Q. For $2 \leq i \leq k$, none of the ways of re-pairing the four vertices u_1, v_1, u_i, v_i can result in more Q-linked pairs than S has. We apply this fact three times. First it follows that no pair in S is a subset of R. We may assume that $u_i \in X$, $1 \leq i \leq k$. Second, by (2) (and an appropriate relabeling of vertices if needed) we may further assume $N_{G_{s,t}}(u_i, Q) = \{q_{i_0}\}$ for $1 \leq i \leq k$. Third, we find that for $1 \leq j \leq m$, we have $v_j \in X$ and $N_{G_{s,t}}(v_j, Q) = \{q_{i_0}\}$.

Let $X' = N_{G_{s,t}}(q_{i_0}, X)$. We have just shown that $\{u_1, \ldots, u_k\} \cup \{v_1, \ldots, v_m\} \subseteq X'$. Therefore

$$|X'| \geq k + m \geq k + t. \tag{3}$$

Observing that $d_G(q_{i_0}, X) \geq d_{G_{s,t}}(q_{i_0}, X) = |X'|$, we have by the choice of $\vec{R}$ and $\vec{Q}$ that

$$d_G(y, X) \geq k + t, \quad \text{for } y \in R \cup \{q_1, q_2, \ldots, q_{i_0}\}. \tag{4}$$

Let $Q' = \{\, q_i \in Q : N_G(q_i, X') \neq \emptyset \,\}$. We now show that

$$\{q_{i_0}\} \subseteq Q' \subseteq \{q_1, q_2, \ldots, q_{i_0}\}. \tag{5}$$

Indeed, suppose that $q_i \in Q'$ for some $i > i_0$. Then there exists $x \in N_G(q_i, X')$. In the $\vec{Q}X\vec{R}$-flip sequence of G, flips of the form $\langle q_{i_0}, x, r\rangle$ (where $r \in R$) are considered before flips of the form $\langle q_i, x, r\rangle$. Therefore $q_{i_0}x \in E(G_{s,t})$ implies $q_i x \in E(G_{s,t})$, which contradicts (2) and proves (5).

Claim 3 We have $i_0 = s$.

In view of (5), it suffices to prove $Q' = Q$. Suppose by way of contradiction that $Q - Q' \neq \emptyset$. Then $(X - X') \cup R \cup Q'$ is a vertex cut in G separating the nonempty sets X' and $Q - Q'$. By connectivity of G and by (3), we have $2k \leq |X \cup R| - |X'| + |Q'| \leq 2k - (k+t) + |Q'|$, so

$$|Q'| \geq k + t \geq 2 + t.$$

By (4), (5) and the above inequality we have

$$e_G(X, Y) \geq e_G(X', Q' \cup R) \geq (k+t)((2+t)+t) > 2k(t+1).$$

On the other hand, using (2) and the fact $X \cup R$ is a clique in $G_{s,t}$, we get

$$e_G(X, Y) = e_{G_{s,t}}(X, Y) = |X|(t+1) < 2k(t+1).$$

This contradiction proves Claim 3.

By counting $E_G(X, Y)$ in two ways we have, by choice of $\vec{Q}$ and $\vec{R}$, that

$$|X|\,(t+1) \geq |Y|\, d_G(q_s, X). \tag{6}$$

By (3) and Claim 3 we have $d_G(q_s, X) \geq k + t > 1 + t$. Therefore $|X| > |Y|$. Alternatively, G is $2k$-connected, so $d_G(q_s, X) \geq 2k - (|Y| - 1)$. Therefore (6) implies

$$(2k - t)(t+1) \geq (t+s)(2k - s - t + 1)$$

$$(s-1)(s - 2k + 2t) \geq 0$$

By the hypothesis, $s - 1 > 0$ so the second factor is non-negative. That is $s + t \geq 2k - t$, which we may write as $|Y| \geq |X|$. This contradicts $|X| > |Y|$ and proves Lemma 6. ■

We now proceed to prove Lemma 2. Let G be a $2k$-connected simple graph with $V(G) = X \cup Y$ where X and Y are disjoint cliques in G. We say that G is *happy* if G contains k Hamilton cycles which are edge-disjoint outside of Y, and that either $|Y| \leq k$ or each of these Hamilton cycles contains an edge in $G[Y]$.

If G has order at most $2k+1$, then by connectivity of G, we have $G = K_{2k+1}$, and G is happy by Proposition 3 (If $|Y| \geq k+1$, then we relabel vertices so that $\{0, 1, \ldots, k\} \subseteq Y$).

We assume G has order at least $2k+2$. Suppose that $Y = \{y\}$. By connectivity we have $|N(y, X)| \geq 2k$. We use Proposition 4 with $H = G - y$, $S = \emptyset$, and k arbitrary pairs in $N(y, X)$, to find k Hamilton paths in $G - y$. These paths extend easily to k edge-disjoint Hamilton cycles in G, so G is happy.

Thus, we assume $|Y| \geq 2$. Suppose $|X| \geq 2k$. Let $X' = \{u_1, v_1, u_2, v_2, \ldots, u_\sigma, v_\sigma\}$ be a maximal subset of X such that each pair $\{u_i, v_i\}$ is Y-linked. If $\sigma < k$, then $|X - X'| \geq 2$, and the graph $G' = G - X'$ satisfies either $d_{G'}(X - X', Y) \leq 1$ or $d_{G'}(Y, X - X') \leq 1$. Therefore G has a cut of size at most $|X'| + 1 < 2k$, a contradiction. Therefore $\sigma \geq k$. We apply Proposition 4 with $H = G[X]$, $S = \emptyset$, and pairs $\{u_i, v_i\}$, $1 \le i \le k$, and then apply Extension 1 (with $Q = Y$) to the resulting paths to obtain k Hamilton cycles in G which are edge-disjoint outside of Y. The cycles produced by Extension 1 always have an edge in $G[Y]$. Therefore G is happy if $|X| \ge 2k$.

We now assume that $1 \le |X| < 2k$, $|V(G)| \ge 2k+2$, and thus $|Y| \ge 3$. By Lemma 6 there exists a partition $Y = Q \cup R$ and orderings $\vec{Q}$, $\vec{R}$ such that $X \cup R$ can be partitioned into k pairs of which at most $|R|-1$ are not Q-linked in $G_{s,t} = G_{s,t}[\vec{Q}X\vec{R}]$, where $s = |Q|$, $t = |R|$. Since G has minimum degree at least $2k = |X \cup R|$, we have that $X \cup R$ is a clique in $G_{s,t}$, and for $x \in X$ we have $d_{G_{s,t}}(x, Q) \ge 1$. Therefore by Lemma 5, the graph $G_{s,t}$ is happy.

It remains to show that if $G' \overset{qXr}{\longmapsto} G''$ is in the $\vec{Q}X\vec{R}$-flip sequence of G, and G'' is happy, then G' is happy. We assume that $q \in Q$, and $r \in R$ are fixed and that $X_{qr} \subseteq X$ is as in (1). We have

$$E(G') - E(G'') = \{qx \mid x \in X_{qr}\} \quad \text{and} \quad E(G'') - E(G') = \{xr \mid x \in X_{qr}\}.$$

Since $Q \cup R$ is a clique in both G' and G'', we have that, for $b \in V(G') - \{r\}$,

$$bq \in E(G'') \quad \text{implies that} \quad bq, br \in E(G') \cap E(G''). \tag{7}$$

Assume that G'' is happy with Hamilton cycles $C_1, \ldots, C_k$. Each C_i is the union of two r,q-paths, so there is a set $\mathcal{P} = \{P_1, \ldots, P_{2k}\}$ of r,q-paths in G'' which are edge-disjoint outside of Y, and a 2-to-1 function $\tau : \mathcal{P} \to \{C_1, \ldots, C_k\}$ such that $C_k = \bigcup \tau^{-1}(C_k)$. (We do not further specify the function τ here, since we will soon be relabeling the paths in $\mathcal{P}$.)

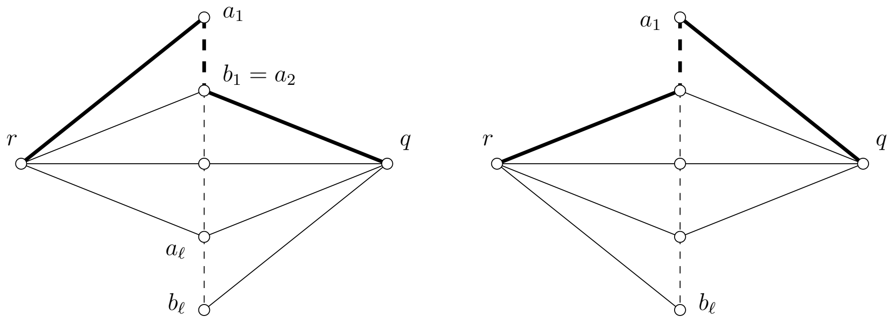
For $1 \le i \le 2k$, let a_i be the neighbour of r in P_i and let b_i be the neighbour of q in P_i. We define an auxiliary directed graph H with $V(H) = \{\, P_i \in \mathcal{P} \mid 1 \le i \le 2k \text{ and } a_i \in X \,\}$, and $\langle P_i, P_j \rangle \in E(H)$ if and only if $b_i = a_j$. Since the paths are edge-disjoint outside of Y, at most one path in $\mathcal{P}$ can use any edge in the set $\{\, qa_j, qb_j, ra_j, rb_j \mid 1 \le j \le 2k \text{ and } b_j \in X \,\}$. Therefore each vertex of H has in-degree and out-degree at most one. Thus each (weak) component of H is a directed path or cycle.

Let $I \subseteq \{1, \ldots, 2k\}$ be the set of indices i such that $a_i \in X_{qr}$. Let $\mathcal{P}_I = \{P_i \in \mathcal{P} \mid i \in I\}$, and consider the subgraph $P = \bigcup_{i=1}^{2k} P_i \subseteq G''$. Then $E(P) - E(G') = \{\, a_i r \mid i \in I \,\}$, and every edge in this set is the first edge of exactly one path in $\mathcal{P}_I$. This correspondence is bijective. For $i \in I$, we have $qa_i \in E(G') - E(G'')$, so the vertex $P_i \in V(H)$ has in-degree zero in H. Let H_i be the weak component of H such that $P_i \in V(H_i)$. Then H_i is a directed path in H, whose initial vertex is P_i. We have shown that the edge $a_i r \in E(P_i)$ is the only edge in the set $\bigcup \{\, E(P_j) \mid P_j \in V(H_i) \,\}$ which does not belong to $E(G')$. Our plan is to modify the paths in $V(H_i)$ so as to eliminate the edge $a_i r$ from this set. After we have performed this modification for each $i \in I$, we shall have a new family of r,q-paths whose edges all belong to G'. We note that if $I = \emptyset$, then $P \subseteq G'$, so G' is happy and there is nothing to prove.

After relabeling paths in $\mathcal{P}$, we may assume that $1 \in I$ and H_1 is the directed path $P_1, P_2, \ldots, P_\ell$. We have that $a_1 \in X_{qr}$ and $b_j = a_{j+1} \in X - X_{qr}$, $j = 1, \ldots, \ell-1$. Since P_ℓ has out-degree zero in H, and since $qb_\ell \in E(P_\ell) \subseteq E(G'')$, we have by (7), that

$$b_\ell \in Y \quad \text{or} \quad b_\ell r \in E(G') - E(P). \tag{8}$$

For $j = 1, \ldots, \ell$, we have $P_j = ra_jR_jb_jq$ where R_j is an a_j, b_j-path in both G'' and G'. The subgraph $\bigcup_{j=1}^{\ell} P_j$ is illustrated in Figure 3 (a). For $j = 1, \ldots, \ell$, let $P'_j = rb_jR'_ja_jq$ where R'_j is



(a) (b)

Figure 3: Diagram (a) shows $\cup_{j=1}^{\ell} P_j$, and (b) shows $\cup_{j=1}^{\ell} P'_j$. The paths P_1 and P'_1 are in bold. The subpaths R_j and R'_j are indicated as dashed lines, $1 \le j \le \ell$.

the reverse of the path R_j. The graph $\bigcup_{j=1}^{\ell} P'_j$ is illustrated in Figure 3 (b). We have

$$E\left(\bigcup_{j=1}^{\ell} P'_j\right) = E\left(\bigcup_{j=1}^{\ell} P_j\right) \cup \{rb_\ell, a_1q\} - \{ra_1, b_\ell q\}. \tag{9}$$

Since $a_1 \in X_{qr}$ and by (8), we have that $\bigcup_{j=1}^{\ell} P'_j \subseteq G'$. Since $qa_1 \notin E(G'')$, we have that $a_1 \neq b_i$ for $1 \le i \le \ell$. It follows that the paths $P'_1, \dots, P'_\ell$ are edge-disjoint outside of Y.

For each $i \in I$ and each $P_j \in V(H_i)$ we define P'_j as we did in the case $i = 1$. We define $P'_m = P_m$ for every $P_m \in \mathcal{P} - \bigcup_{i \in I} V(H_i)$. For $h = 1, \dots, k$, we define $C'_h = P'_j \cup P'_m$, where $\tau^{-1}(C_h) = \{P_j, P_m\}$. (The function τ is defined near (7).) Since $V(P'_j) = V(P_j)$ ($j = 1, \dots, 2k$), each C'_h is a Hamilton cycle in G'. Because the paths $P'_1, \dots, P'_k$ are edge-disjoint outside of Y, the same is true for the cycles $C'_1, \dots, C'_k$. To conclude that G' is happy, it suffices to show that if C_h contains an edge in $G''[Y]$, then C'_h contains an edge in $G'[Y]$ $(= G''[Y])$. Referring Figure 3, we see that every edge $e \in E(C_h) - E(C'_h)$ has either has some vertex $a_j \in X$ as an endpoint, or has b_ℓ as an endpoint. If $b_\ell \in X$, then e is not a edge of $G''[Y]$. If $b_\ell \in Y$, then $b_\ell q$ is an edge of C'_h belonging to $G'[Y]$. Therefore G' is happy, and Lemma 2 is proved. ■

4 Proof of Theorem 1

Suppose by way of contradiction that there exists a simple undirected graph G, a subset $W \subseteq V(G)$, and an integer k such that the triple $\langle G, W, k\rangle$ satisfies the hypothesis, but not the conclusion of Theorem 1. We may assume that $E(G[W])$ is maximal. That is, for each pair of non-adjacent vertices $u, v \in W$, the graph $G + uv$ either has k pairwise edge-disjoint cycles through W, or the triple $\langle G + uv, W, k\rangle$ does not satisfy the hypothesis of Theorem 1. Let

$$Y = \{\, v \in W \mid d_G(v) \ge \frac{n}{2} + 2(k-1)\,\}.$$

Since $G[W]$ is $2k$-connected, we have $|W| \ge 2k+1$. By Proposition 3, $G[W]$ is not complete, and hence $Y \neq \emptyset$ by the hypothesis.

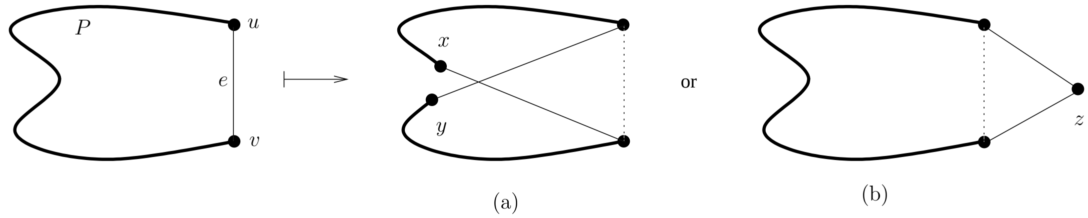


Figure 4: Two ways to eliminate the edge $e = uv$ from $C_1 = P + e$.

Claim 1 If G has k cycles through W which are edge-disjoint outside of Y, then

a) G has k pairwise edge-disjoint cycles through W.

b) If, moreover, for some $uv \in E(G[Y])$ $d_G(u)$, $d_G(v) \geq \frac{n}{2} + 2k - 1$, then $G - uv$ has k pairwise edge-disjoint cycles through W.

Proof of part a). Let $C_1, \dots, C_k$ be cycles through W in G which are edge-disjoint outside of Y, and such that

$$p = \sum_{1 \leq i < j \leq k} |E(C_i) \cap E(C_j)|$$

is as small as possible. Suppose by way of contradiction that $p > 0$. Without loss of generality, there exists $uv \in E(C_1) \cap E(C_2) \subseteq E(G[Y])$. Let $P = C_1 - uv$, and let

$$G' = \left(G - \bigcup_{i=1}^{k} E(C_i)\right) + E(P).$$

By definition of Y, and since $uv \in E(C_2)$ we have

$$d_{G'}(u) + d_{G'}(v) \geq d_G(u) + d_G(v) - 4(k-1) + 2 \geq n + 2. \tag{10}$$

It follows that either $d_{G'}(u, V(P)) + d_{G'}(v, V(P)) \geq |V(P)| + 2$ or $d_{G'}(u, V(G) - V(P)) + d_{G'}(v, V(G) - V(P)) \geq n - |V(P)| + 1$. In the former case, there exist consecutive vertices x, y along the u, v-path P such that $uy, vx \in E(G') \subseteq E(G)$, and we define $D_1 = C_1 - \{uv, xy\} + \{uy, vx\}$ (see Figure 4 (a)). In the latter case, there exists $z \in V(G) - V(P)$ such that $uz, vz \in E(G)$, and we let $D_1 = C_1 - \{uv\} + \{uz, vz\}$ (see Figure 4 (b)). In both cases, D_1 is a cycle in G which goes through W. Let $D_i = C_i$ for $i = 2, \dots, k$. Now $D_1, \dots, D_k$ are cycles which satisfy the assumptions of the claim with $\sum_{i \neq j} |E(D_i) \cap E(D_j)| = p - 1$, a contradiction. Therefore $p = 0$ and $C_1, \dots, C_k$ are pairwise edge-disjoint cycles in G.

Proof of part b). Let $uv \in E(G[Y])$ so that $d_G(u)$, $d_G(v) \geq \frac{n}{2} + 2k - 1$. We may assume that all cycles are edge-disjoint by part a). Now, assume without loss of generality that $uv \in E(C_1)$. We can repeat the above procedure except that now we cannot use the fact that $uv \in E(C_2)$ to provide the term "+2" in (10). Instead we rely on the slightly stronger lower bound on $d_G(u)$ and $d_G(v)$ to recover inequality (10). Thus, we can modify C_1 so that it will not contain the edge uv.

Claim 2 The graph $G[Y]$ is complete.

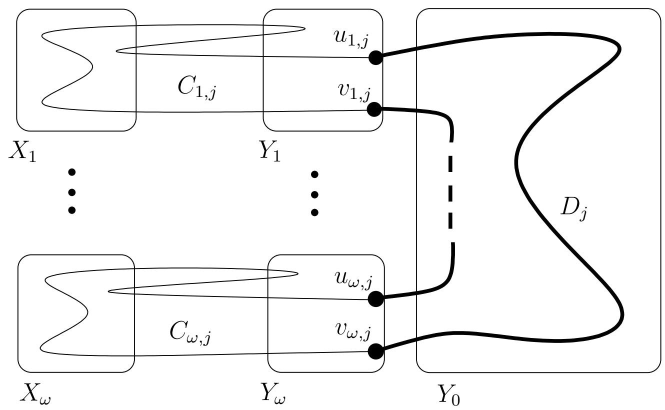


Figure 5: Constructing the Hamilton cycle C_j of $G[W]$ from the cycles D_j (in bold), and $C_{i,j}$, $1 \le i \le \omega$.

Suppose that $xy \notin E(G)$ for some $x, y \in Y$. Let $G' = G + xy$. If $u, v \in W$ satisfy $\mathrm{dist}_{G'[W]}(u,v) = 2$ and $\mathrm{dist}_{G[W]}(u,v) \neq 2$, then either u or v belongs to $\{x, y\} \subseteq Y$. Therefore G' satisfies the hypothesis of Theorem 1. By the choice of G, the graph G' has k pairwise edge-disjoint cycles through W. Using Claim 1b, these cycles can be modified so that they avoid the edge xy. This contradicts that G is a counterexample, and proves Claim 2.

Let $X = W - Y$. By Claim 2, Proposition 3, and the fact that G is a counterexample, $X \neq \emptyset$. Let $G_i = (X_i, E_i)$, $1 \le i \le \omega$, be the connected components of $G[X]$, for some $\omega \ge 1$. Let $Y_i = N_G(X_i, Y)$, $1 \le i \le \omega$. By the definition of Y, no pair of vertices of X is at distance two in $G[W]$. Consequently, G_i is complete and $Y_i \cap Y_j = \emptyset$ for $1 \le i < j \le \omega$. Let $Y_0 = Y - \cup_{i=1}^{\omega} Y_i$. Then $W = X \cup Y_0 \cup Y_1 \cup \cdots \cup Y_\omega$.

Claim 3 $|Y_i| \ge 2k$*, for* $i = 1, \ldots, \omega$.

Suppose that $|Y_i| < 2k$ for some $1 \le i \le \omega$. Since $G[W]$ is $2k$-connected, it follows that $\omega = 1$ and $Y = Y_1$. Hence by Lemma 2, $G[W]$ has k Hamilton cycles which are edge-disjoint outside of Y, and if $|Y| \ge k+1$, then each of them contains an edge in $G[Y]$. This, together with Claim 1a, contradicts that G is a counterexample.

Claim 4 The graph $G[W]$ *has* k *Hamilton cycles* $C_1, \ldots, C_k$ *which are edge-disjoint outside of* Y.

Let $i \in \{1, \ldots, \omega\}$. Since $G[W]$ is $2k$-connected, and $G[X_i], G[Y_i]$ are complete, and $E_G(X_i, W - X_i) = E_G(X_i, Y_i)$, the graph $G[X_i \cup Y_i]$ is $2k$-connected. By Claim 3, $|Y_i| \ge 2k \ge k+1$, and by Lemma 2 the graph $G[X_i \cup Y_i]$ has k Hamilton cycles $C_{i,1}, \ldots, C_{i,k}$ which are edge-disjoint outside of Y_i, and such that each $C_{i,j}$ ($1 \le j \le k$) contains an edge, say $u_{i,j}v_{i,j}$ in $G[Y_i]$.

Recall that $W = X \cup Y_0 \cup Y_1 \cup \cdots \cup Y_\omega$. For each $j \in \{1, \ldots, k\}$ we construct a Hamilton cycle C_j of $G[W]$ as follows. The complete graph $G[Y_0 \cup_{i=1}^{\omega} \{u_{i,j}, v_{i,j}\}]$ is either the single edge $u_{1,j}v_{1,j}$, or it has a Hamilton cycle D_j passing through all the edges in $\{\, u_{i,j}v_{i,j} \mid 1 \le i \le \omega \,\}$. In the former case, we define $C_j = C_{1,j}$. In the latter case we obtain C_j from D_j by replacing each edge $u_{i,j}v_{i,j} \in E(D_j)$ by the path $C_{i,j} - u_{i,j}v_{i,j}$, ($1 \le i \le \omega$). See Figure 5. In either case, C_j is a Hamilton cycle of $G[W]$. Since the cycles $C_{i,j}$ are edge-disjoint outside of Y, the same is true for the cycles $C_1, \ldots, C_k$. This proves Claim 4.

Theorem 1 now follows from Claim 1a. ∎

Remark 7 The cycles constructed in the proof of Theorem 1 use no edges in $E(G - W)$. This is reflected in the fact that a triple $\langle G, W, k\rangle$ satisfies the hypothesis of Theorem 1 if and only if $\langle G - E(G - W), W, k\rangle$ satisfies the hypothesis of Theorem 1.

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References

[1] Béla Bollobás, Graham Brightwell, Cycles through specified vertices, *Combinatorica* **13** (1993), 147–155.

[2] Genghua Fan, New sufficient conditions for cycles in graphs, *J. Combin. Theory Ser. B* **37** (1984), 221–227.

[3] R. J. Faudree, C. C. Rousseau, R. H. Schelp, Edge-disjoint Hamiltonian cycles, *Graph theory with applications to algorithms and computer science (Kalamazoo, Mich., 1984)*, Wiley-Intersci. Publ., (1985), 231–249.

[4] Roland J. Gould, Updating the Hamiltonian problem–a survey, *J. Graph Theory* **15** (1991), 121–157.

[5] Roland J. Gould, Advances on the Hamiltonian problem—a survey, *Graphs Combin.* **19** (2003), 7–52.

[6] Guojun Li, Edge disjoint Hamilton cycles in graphs, *J. Graph Theory* **35** (2000), 8–20.

[7] Guojun Li, Chuanping Chen, Disjoint Hamiltonian cycles in graphs, *Australas. J. Combin.* **19** (1999), 83–89.

[8] D. E. Lucas, Recreations mathematiques, Vol. II, Gauthiers Villars, Paris, 1892.