Triangular embeddings of complete graphs from graceful labellings of paths

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Abstract

We show that to each graceful labelling of a path on $2s+1$ vertices, $s \geq 2$, there corresponds a current assignment on a 3-valent graph which generates at least 2^{2s} cyclic oriented triangular embeddings of a complete graph on $12s+7$ vertices. We also show that in this correspondence, two distinct graceful labellings never give isomorphic oriented embeddings. Since the number of graceful labellings of paths on $2s+1$ vertices grows asymptotically at least as fast as $(5/3)^{2s}$, this method gives at least 11^s distinct cyclic oriented triangular embedding of a complete graph of order $12s+7$ for all sufficiently large s.

1 Introduction

The dual version of the famous and more than 100 years old Map Colouring Problem is to determine the smallest genus of an orientable (and a nonorientable) surface on which a given complete graph embeds. The problem remained unsolved until the late 1960's work of Ringel, Youngs and others [8]. The main tool of the solution were 'current graphs' used to generate rotation systems of embeddings. In the case of embeddings on orientable surfaces, rotation systems are in a one-to-one correspondence with *oriented* embeddings, those with a preassigned orientation of the supporting surface. Soon it was

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realized that the methods of Ringel et al. are, in fact, covering construction techniques in disguise; we recommend [5] for more history and details.

In order to illustrate the above we focus on constructions of oriented triangular (and hence smallest genus) embeddings of the complete graph K_{12s+7}. In a more up-to-date language, a 'current graph' giving such an embedding is a connected, trivalent graph on $4s + 2$ vertices, embedded on an oriented surface with a single face. We assume that each edge of the current graph has a preassigned direction and that a 'current' with values in Z_{12s+7} flows along the edge (with the convention that the inverse of the current flows in the opposite direction). The current assignment is assumed to satisfy the *Kirchhoff's Current Law*, which is to say that the sum of the incoming currents is equal to the sum of the outgoing currents at each vertex. The final assumption is that the set of all $(+/-)$ currents is equal to the set of non-zero elements of Z_{12s+7}. Methods of topological graph theory then allow one to 'transform' such a current graph into a *cyclic* oriented triangular embedding of K_{12s+7}, that is, an embedding in which Z_{12s+7} acts regularly on vertices as a group of orientation-preserving automorphisms of the embedding.

Several papers have recently addressed lower bounds on the number of (pairwise non-isomorphic) oriented triangular embeddings of complete graphs. The current record holders are surgical constructions of [3] and [4] where it is shown that the number of triangular embeddings of K_n is bounded below by 2^{cn^2} for n in certain residue classes of some multiples of 12. These embeddings, however, are far from cyclic. *Nota bene*, a generous upper bound on the number of oriented cyclic embeddings of K_n is $(n-2)!$ regardless of the face lengths, which can be obtained by assigning an arbitrary rotation to one vertex and carrying the rotation over to all other vertices by the cyclic action. (This is in a sharp contrast with the upper bound $(n-2)!)^n$ on the number of arbitrary oriented embeddings of K_n.) Going back to our special case $n = 12s + 7$, in order to obtain a large number of *cyclic* oriented triangular embeddings of K_{12s+7} the above topological construction offers three freedoms of choice: The choice of the trivalent graph orientably embeddable with a single face, the choice of its single-face embedding, and the choice of the current assignment. The first possibility has not yet been considered in the literature. In fact, there seem to be just two trivalent graphs used in the past: The Ringel graph [8] which we will use in Section 2 and the Youngs graph [10]. The second possibility was exploited, for the Ringel graph, in [6] to show that K_{12s+7} has at least 4^s non-isomorphic cyclic oriented triangular embeddings.

In this paper we show that the choice of current assignment can also be used to produce an exponential number of cyclic embeddings of K_{12s+7}. Surprisingly, such current assignments (and hence the cyclic embeddings) can be obtained from graceful labellings of paths. We prove that, for $s \geq 2$, to each graceful labelling of a path on $2s+1$ vertices there corresponds a current assignment on a 3-valent graph that gives at least 4^s pairwise non-isomorphic cyclic oriented triangular embeddings of K_{12s+7}. Moreover, we show that this correspondence has the property that two distinct graceful labellings never give isomorphic oriented embeddings. Our ingredients are a construction of an exponential number of graceful labellings of paths [1] and substantial results of [6] and [7] on generating triangular embeddings of complete graphs from current graphs and distinguishing embeddings that arise from different current assignments.

2 The result

Let G be a connected graph with n vertices and m edges. A *labelling* of G is any one-to-one function f that assigns integers between 1 and $m+1$ to vertices of G. For each edge $e = uv$ the value $f(e) = |f(u) - f(v)|$ is the *induced edge-label*; using the same symbol f for edge-labels will cause no confusion. The labelling f is said to be *graceful* if the set of distinct induced edge-labels coincides with the set $\{1, 2, \ldots, m\}$. Such labellings are of particular interest for trees. By the famous and still unresolved Graceful Tree Conjecture, each tree has a graceful labelling. So far only a few infinite classes of trees have been proved to have graceful labellings. Here we will be interested only in paths, which trivially have graceful labellings. In fact, by [1] the number of such labellings of a path of length n grows exponentially with n.

The surprising connection between graceful labellings of paths and triangular embeddings of complete graphs can be formally presented as follows.

Theorem 1 *To each graceful labelling f of a path on $2s+1$ vertices, $s \geq 2$, there corresponds a collection $\mathcal{C}_f$ of 4^s cyclic oriented triangular embeddings of a complete graph on $12s+7$ vertices. Moreover, if f and g are distinct graceful labellings of a path on $2s+1$ vertices, then no oriented embedding in $\mathcal{C}_f$ is isomorphic to an oriented embedding in $\mathcal{C}_g$.*

PROOF. Let P be a path on $2s+1$ vertices and let f be a graceful labelling of P, so that f is a bijection from the vertex set of P onto the set of labels $\{1, 2, \ldots, 2s+1\}$. Let H be a supergraph of P obtained by adding two new vertices b and t together with the edge bt and the $4s+2$ edges of the form bu and ut where u ranges over all vertices of P. The graph H triangulates the plane; one may think of H as being embedded so that b and t are the 'bottom' and 'top' vertices while P shows up as a 'horizontal' path; see the part of Fig. 1 drawn in solid lines. We transform f to a function f' on the vertex set of H defined by $f'(b) = 1$, $f'(t) = 6s+4$, and $f'(u) = 3f(u)$ for each vertex u of P. Observe that f' is a graceful labelling of the graph H, since the induced edge-labels on the $2s$ edges of P and on the edge bt, on the $2s+1$ edges bu, and on the $2s+1$ edges ut, coincide with the sets of integers between 1 and $6s+3$ which are congruent to 0, 2, and 1, respectively.

Let H^* be the dual graph to the (planar) graph H, shown in Fig. 1 in dashed lines. The graph H^* is, in fact, the 3-valent Ringel graph that underlies the current graph construction of a cyclic orientable triangular embedding of K_{12s+7} given in Section 2.3 of [8]. We now transform the edge-labels of the graceful labelling f' of H into a current assignment on H^* as follows. Let us direct each edge of H from the vertex carrying the smaller value of f' to the vertex with the larger value. To describe the induced edge directions of the dual graph H^* let us first fix a clockwise orientation of every face of the plane embedding of H. We now direct every edge e^* of H^* *towards* the face whose orientation on the face boundary agrees with the orientation of e. Finally, for any directed edge e of H, the current $f^*(e^*)$ on the directed edge e^* of H^* dual to e is defined to be equal to $f'(e)$.

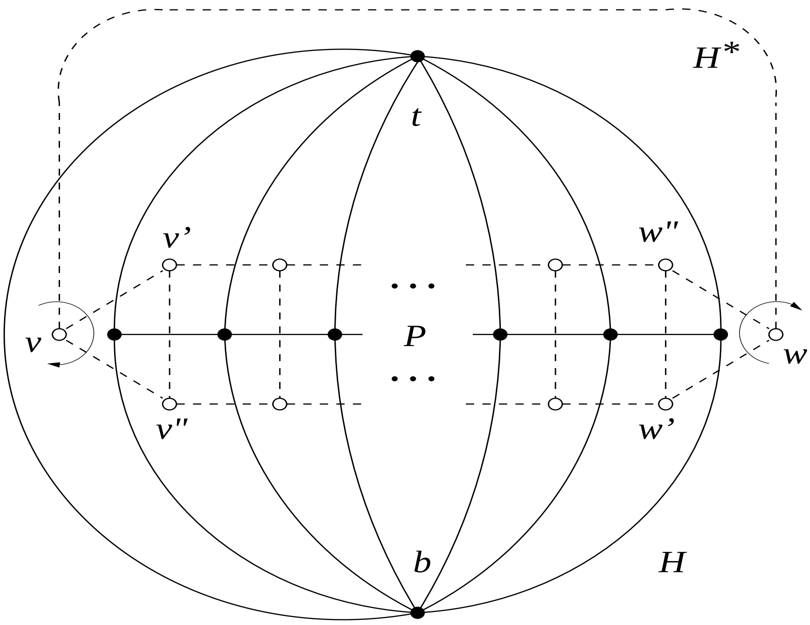

Fig. 1. The graph H and its dual H^*, the Ringel graph.

Consider a triangle with vertices u, v, w in the planar embedding of H. The way the labelling f' was introduced implies that we may always choose the notation for the three vertices such that the boundary of the triangle consists of the directed path uvw from u to w and the directed edge uw from u to w. Moreover, with this notation we have $f'(uv) + f'(vw) = f'(uw)$ for any triangular face of H. It is easy to see that the above translates into the 'dual language' as follows. At each vertex of H^* we have either two incoming dual directed edges and one outgoing edge, or vice versa. In the first case, the sum of the currents on the incoming dual edges is equal to the current on the outgoing dual edge; the situation in the second case is analogous. We conclude that the current assignment f^* satisfies the Kirchhoff's Current Law at each vertex of H^*.

So far, f' and f^* have been treated as integer-valued functions. From now on we assume that the range of both f' and f^* is the group Z_{12s+7}. Clearly, all the above calculations and conclusions remain valid in this new setting.

Following the topological background reviewed briefly in the Introduction, we now want to embed H^* in an oriented surface with a single face. It is known (cf. [5]) that all such oriented embeddings are in a one-to-one correspondence with certain permutations which we shall for convenience call 1*-rotation systems*. To describe these, consider each edge of H^* to be formed by a pair of oppositely directed edges (often called *arcs*). Let A be the set of arcs of H^*; it follows that $|A|$ is equal to twice the number of edges of H^*, Let θ be the involution on A assigning to every arc its reverse. A 1-rotation system for H^* is a permutation π of A such that (1) each cycle of π is a cyclic permutation of arcs emanating from a vertex of H^*, and (2) the permutation $\pi\theta$ is cyclic and of order $|A|$.

Next, we use results of [6] to describe certain set of single-face embeddings. Our starting point is the observation that the 1-rotation system for the first graph in Fig. 2 of [6] (which is the graph in Fig 1 for $s = 1$) where both v and w receive clockwise 'local rotations' can easily be extended to the graphs H^* for any $s \geq 2$. It follows that there exists a 1-rotation system π for H^* such that the cycles of π at the vertices v and w are (vw, vv_1, vv_2) and (wv, ww_1, ww_2), respectively, as indicated by the arrows around v and w in Fig. 1. By Lemma 5 of [6], there exists a set $\mathcal{S}$ of 2^{2s} different 1-rotation systems σ for H^* whose restrictions to arcs emanating from v and w coincide with π. As we know from [8] or [5], for each $\sigma \in \mathcal{S}$ the pair (f^*, σ) determines a cyclic oriented triangular embedding of K_{12s+7}. Recalling that the current assignment f^* arose from a graceful labelling f of the path P, we let $\mathcal{C}_f$ denote the collection of all oriented embeddings derived from the 2^{2s} pairs (f^*, σ), where $\sigma \in \mathcal{S}$.

To finish the proof of the theorem it remains to show that for any two graceful labellings f and g of P and for any two distinct 1-rotation systems $\sigma, \tau \in \mathcal{S}$, the pairs (f^*, σ) and (g^*, τ) determine isomorphic oriented embeddings if and only if $f = g$ and $\sigma = \tau$ (which also implies that each collection $\mathcal{C}_f$ contains 2^{2s} pairwise non-isomorphic oriented embeddings). Fortunately, here we can rely on a profound analysis of isomorphism of such embeddings, carried out in [6] for a fixed current assignment and in [7] for arbitrary assignments. The most general result about distinguishing embeddings arising from current graphs is Theorem 2 of [7], dealing with unoriented embeddings that comprise nonorientable embeddings and orientable embeddings with unspecified orientation of the surface. Making use of our convention that pairs of reverse arcs carry opposite currents, the variation of Theorem 2 of [7] for oriented embeddings that applies to our setting reads as follows:

The pairs (f^*, σ) *and* (g^*, τ) *determine isomorphic oriented embeddings of* K_{12s+7} *if and only if the graph* H^* *admits an automorphism* φ *(regarded as a permutation of the arc set* A*) and the group* Z_{12s+7} *admits an automorphism* α *such that* $\varphi\sigma(e) = \tau\varphi(e)$ *and* $g^*\varphi(e) = \alpha f^*(e)$ *for every arc* $e \in A$*.*

Suppose now that (f^*, σ) and (g^*, τ) determine isomorphic oriented embeddings of K_{12s+7} where $s \geq 2$ and let φ and α be as in the above statement. The graph H^* then has just four automorphisms which, in Fig. 1, may be identified as the identity, the flips across the horizontal and the vertical axis of symmetry, and the central rotation by 180^o. If φ is one of the two flips, then the condition $\varphi\sigma(e) = \tau\varphi(e)$ is violated at arcs at v or w, since σ and τ agree with the 1-rotation π on the vertices v and w by definition. Let φ be the central rotation; observe that for the arc wv we have $\varphi(wv) = vw$. Since the arc wv receives the current $6s+3$ in *all* the current assignments derived from graceful labellings of P, from $g^*\varphi = \alpha f^*$ applied to $e = wv$ we obtain $\alpha(6s+3) = 6s+4$. The unique such automorphism of Z_{12s+7} is $\alpha(j) = -j$ for all $j \in Z_{12s+7}$. Using this information about α and applying the previous identity to the arc v_1v (see Fig. 1) gives $g^*(w_1w) = g^*\varphi(v_1v) = -f^*(v_1v) = f^*(vv_1)$. This, however, contradicts the fact that *all* our current assignments derived from graceful labellings of P assign to the arcs vv_1 and w_1v a positive integer smaller than $6s+3$ and congruent to 1 and 2, respectively. Finally, if φ is the identity, then α must be the identity as well, and hence $\sigma = \tau$ and $f^* = g^*$. This completes the proof. □

3 Remarks

Let $\gamma(n)$ denote the number of graceful labellings of a path on n vertices. In [1] it was proved that $\gamma(n) > (5/3)^n$ for all sufficiently large n. The number of graceful labellings on the path P considered in the previous proof is therefore certainly larger than $(5/3)^{2s}$ for all sufficiently large s. Since $4^s(5/3)^{2s} > 11^s$, we have the following consequence of our main result.

Corollary 2 *For any* $s \geq 2$ *there are at least* $4^s\gamma(2s)$ *pairwise non-isomorphic oriented triangular embeddings of* K_{12s+7}*. In particular, for any sufficiently large* s *the number of such embeddings is larger than* 11^s*.* □

As mentioned in the Introduction, to construct a large number of cyclic oriented triangulations by complete graphs treher are only three choices: Vary the single-face embeddings of a suitable current graph, vary the current assignment, or vary the current graph itself (which has not yet been considered). Our approach combines the first two choices. Moreover, since computational evidence of [1] suggests that the number of graceful labellings of a path on n vertices is at least $(cn)^n$ for some $c > 0$, our main result appears to have a far better potential for generating large numbers of complete triangulations than any of the previously used methods.

It should be pointed out that a similar potential as regards the order of magnitude lies in trying to work with different trivalent graphs. The first obvious candidates are the $(2s)!$ graphs one obtains by varying the Ringel-Youngs graph H^* by differently reattaching the 'bottom rungs' of the ladder, which certainly gives more than $(cs)^{2s}$ nonisomorphic graphs for some $c > 0$. It is not clear, however, whether such graphs admit suitable current assignments.

For completeness, we note that graphs admitting an orientation and a current assignment as observed on H^* have appeared in the literature under the name *conservative graphs* [2]. A topological interpretation of conservative graphs that agrees with what has been presented in Secion 2 was given in [9].

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