PACKING NON-ZERO A-PATHS IN GROUP-LABELED GRAPHS

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ABSTRACT. Let $G = (V, E)$ be an oriented graph whose edges are labeled by the elements of a group Γ and let $A \subseteq V$. An *A-path* is a path whose ends are both in A. The *weight* of a path P in G is the sum of the group values on forward oriented arcs minus the sum of the backward oriented arcs in P. (If Γ is not abelian, we sum the labels in their order along the path.) We are interested in the maximum number of vertex-disjoint A-paths each of non-zero weight. When $A = V$ this problem is equivalent to the maximum matching problem. The general case also includes Mader's $\mathcal{S}$-paths problem. We prove that for any positive integer k, either there are k vertex-disjoint A-paths each of non-zero weight, or there is a set of at most $2k - 2$ vertices that meets each of the non-zero A-paths. This result is obtained as a consequence of an exact min-max theorem.

1. INTRODUCTION

Let Γ be a group, let $G = (V, E)$ be an oriented graph where each edge e of G is assigned a weight $\gamma_e \in \Gamma$, and let $A \subseteq V$. (We will use additive notation for groups, although they need not be abelian.) An *A-path* is a path (with at least one edge) in the underlying graph whose ends are both in A. Let e be an edge of G oriented with tail u and head v. We let $\gamma(e, u) = -\gamma_e$ and $\gamma(e, v) = \gamma_e$. Now, if $P = (v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ is a path in G, then the *weight of P*, denoted $\gamma(P)$, is defined to be $\sum_{i=1}^{k} \gamma(e_i, v_i)$. Note that, reversing the orientation on an edge e and replacing γ_e with $-\gamma_e$ does not change the weight of any path.

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We are interested in the maximum number of vertex-disjoint A-paths each of non-zero weight. We prove the following result.

Theorem 1.1. *Let Γ be a group, let $G = (V, E)$ be an oriented graph with edge-labels from Γ, and let $A \subseteq V$. Then, for any integer k, either*

(1) *there are k vertex-disjoint A-paths each of non-zero weight, or*
(2) *there is a set of at most $2k-2$ vertices that meets each non-zero A-path.*

Let $\nu(G, A, \gamma)$ denote the the maximum number of vertex-disjoint A-paths each of non-zero weight. We prove Theorem 1.1 as a corollary to an exact min-max theorem for $\nu(G, A, \gamma)$. In fact we will give two different versions of the min-max theorem; the first provides a more intuitive upper-bound while the second is cleaner. Let $E(A, \gamma)$ denote the set of all edges $e \in E$ whose ends are both in A and that have $\gamma_e = 0$; note that deleting such edges does not affect ν. Let $\text{comp}(G)$ denote the set of components of G. Finally, let $X, A' \subseteq V$ such that $A - X \subseteq A' \subseteq V - X$ and let $G' = G - X - E(A', \gamma)$. Then

$$\begin{aligned}
\nu(G, A, \gamma) &\leq |X| + \nu(G - X, A - X, \gamma) \\
&\leq |X| + \nu(G - X, A', \gamma) \\
&= |X| + \nu(G', A', \gamma) \\
&= |X| + \sum_{H \in \text{comp}(G')} \nu(H, A' \cap V(H), \gamma) \\
&\leq |X| + \sum_{H \in \text{comp}(G')} \left\lfloor \frac{|A' \cap V(H)|}{2} \right\rfloor .
\end{aligned}$$

We will see that after an appropriate change of edge-weights we can find X and A' such that the above inequalities hold with equality.

Let $x \in V$ and let $\delta \in \Gamma$. For each edge e of G with tail u and head v we define

$$\gamma'_e = \begin{cases} \gamma_e + \delta, & \text{if } v = x \\ -\delta + \gamma_e, & \text{if } u = x \\ \gamma_e, & \text{otherwise.} \end{cases}$$

We say that γ' is obtained by *shifting* γ by δ at x. Note that, if $x \notin A$ then this shift does not change the weight of any A-path (even when Γ is non-abelian). The main theorem is:

Theorem 1.2. *Let Γ be a group, let $G = (V, E)$ be an oriented graph with edge labels $(\gamma_e : e \in E)$ from Γ, and let $A \subseteq X$. Then there exist edge-labels $(\gamma'_e : e \in E)$ obtained by shifting γ at vertices in $V - A$ and* *there exist sets* $X, A' \subseteq V$ *such that* $A - X \subseteq A' \subseteq V - X$ *and*

$$\nu(G,A,\gamma) = |X| + \sum_{H \in \text{comp}(G')} \left\lfloor \frac{|A' \cap V(H)|}{2} \right\rfloor,$$

where $G' = \text{comp}(G - X - E(A', \gamma'))$.

We now turn to an alternative min-max theorem. A set of edges $F \subseteq E$ is *A-balanced* if F contains no non-zero A-path and no non-zero circuit. We let $V(F)$ denote the set of all vertices in G that are incident with an edge in F. It is straightforward to prove that $F \subseteq E$ is A-balanced if and only if there exist edge-labels $(\gamma'_e : e \in E)$ obtained by shifting γ at vertices in $V - A$ such that $\gamma'_f = 0$ for all $f \in F$. With this in mind, the next result is an easy consequence of Theorem 1.2.

Corollary 1.3. *Let* Γ *be a group, let* $G = (V, E)$ *be an oriented graph with edge labels* $(\gamma_e : e \in E)$ *from* Γ*, and let* $A \subseteq X$*. Then*

$$\nu(G,A,\gamma) = \min \left(|X| + \sum_{H \in \text{comp}(G-X-F)} \left\lfloor \frac{|(A \cup V(F)) \cap V(H)|}{2} \right\rfloor \right),$$

where the minimum is taken over all A-balanced sets $F \subseteq E$ *and all sets* $X \subseteq V$.

Note that, $\nu(G, V, \gamma)$ is the size of the largest matching in $G - E(V, \gamma)$. When $A = V$ it is easy to see that Theorem 1.2 is equivalent to the Tutte-Berge Formula (Theorem 3.1) for the size of a maximum matching. Our proof of Theorem 1.2 is modeled on an easy proof of the Tutte-Berge Formula that we give in Section 3.

2. Some special cases

In this section we mention some path-packing problems that can be modeled via non-zero A-paths. In each of the applications we are given an undirected graph $G = (V, E)$ and a set $A \subseteq V$. Then, we are interested in finding a maximum collection of "feasible" A-paths; where feasibility depends on the application. We then determine a group, a labeling of the edges, and an orientation of G so that the non-zero A-paths and feasible A-paths coincide. Unless explicitly defined, we assume that an arbitrary orientation of G has been prescribed.

A-paths. Here we consider any A-path to be feasible. We assign labels γ_e to edges $e \in E$ and let Γ be the free group generated by $\{\gamma_e : e \in E\}$. Thus, any non-trivial path has non-zero weight. Gallai [2] reduced this case to the maximum cardinality matching problem and proved the specialization of Theorem 1.1.

Odd A-paths. Here only the A-paths of odd length are feasible. We let $\Gamma = \mathbb{Z}_2$ and assign to each edge the label 1. Thus the non-zero paths are exactly those of odd length. The problem of finding a maximum collection of disjoint odd A-paths can be reduced to the maximum matching problem; see [1].

(The main result in [1] gives a structural characterization of signed-graphs with no odd-K_n minor. Signed-graphs can be considered as binary coextensions of graphic matroids. Theorem 1.1 allows us to extend those results to coextensions of graphic matroids over other finite fields.)

(S,T)-paths. Let (S,T) be a partition of A; an (S,T)-path is a path with one end in S and the other end in T. Let $\Gamma = \mathbb{Z}_2$. The edges with exactly one end in S are assigned a label of 1 and all other edges are labeled 0. Then, an A-path is an (S,T)-path if and only if it is non-zero. Now, $\nu(G,A,\gamma)$ is just the maximum number of vertex disjoint (S,T)-paths. It is an interesting exercise to deduce Menger's theorem from Theorem 1.2.

Composition of feasible sets. Suppose that we have groups Γ_1 and Γ_2 and two edge-labelings $(\alpha_e : e \in E)$ from Γ_1 and $(\beta_e : e \in E)$ from Γ_2. We can define $\Gamma = \Gamma_1 \times \Gamma_2$ and define new edge-labels $\gamma_e = (\alpha_e, \beta_e)$. Now, a path P is non-zero with respect to γ if and only if P is non-zero with respect to either α or β.

Mader's $\mathcal{S}$-paths. Let $\mathcal{S}$ be a partition of A and let $l = |S|$. A path is an $\mathcal{S}$-path if its ends are in different parts of $\mathcal{S}$. Thus, an A-path is an $\mathcal{S}$-path if and only if it is an $(S, A-S)$-path for some set $S \in \mathcal{S}$. Then, by composition, we can define a group $\Gamma = \mathbb{Z}_2^l$ and an edge-labeling γ from Γ such that the $\mathcal{S}$-paths are precisely the non-zero A-paths. (There is a more direct formulation in which $\Gamma = \mathbb{Z}_l$.) Mader [3] proved a min-max theorem for the maximum number of disjoint $\mathcal{S}$-paths; see Schrijver [5] for a shorter proof. Mader's Theorem is a direct specialization of Corollary 1.3.

(The problem of finding a maximum collection of vertex-disjoint $\mathcal{S}$-paths is equivalent to the problem of finding a maximum collection of internally vertex-disjoint A-paths. It is natural then to consider the problem of finding a maximum collection of internally vertex-disjoint non-zero A-paths. This contains the problem of finding a maximum collection of internally vertex-disjoint odd paths between a given pair of vertices; we suspect that this latter problem is NP-hard.)

Paths on surfaces. Suppose that $G = (V,E)$ is an oriented graph embedded on a surface S and that $A \subseteq V$ all lie on a common face F

in the embedding, where F is a closed disk. We fix a basepoint x in F; then, we associate to each A-path P a simple closed curve $C(P)$ on S that is contained in $P \cup F$ and that has x as its basepoint. Now, we can designate an A-path P to be feasible, in different ways, according to the homotopy class of $C(P)$.

Example 1: P is feasible if $C(P)$ is non-contractible.
Example 2: P is feasible if $C(P)$ is non-separating.
Example 3: P is feasible if $C(P)$ is orientation reversing (that is, the neighbourhood of the curve $C(P)$ is a Möbius band).

Let $\Gamma = \pi(S, x)$ be the fundamental group of S with respect to the basepoint x; see Munkres [4]. Recall that the elements of Γ are the equivalence classes of (x, x)-curves on S with respect to homotopy; thus, the identity of Γ consists of the set of contractible (x, x)-curves. Readers familiar with topology will see that:

Lemma 2.1. *G can be assigned edge-labels $(\gamma_e : e \in E)$ from Γ such that, for any A-path P, $\gamma(P)$ is the homotopy class of $C(P)$.*

Thus, given the edge-labeling γ from Lemma 2.1, an A-path P is non-zero if and only if $C(P)$ is non-contractible. This gives us a formulation for the first example. In each of the other two examples, the homotopy classes corresponding to non-feasible A-paths determine a normal subgroup of Γ. Therefore, formulations for these examples can be obtained, via Lemma 2.1, by applying appropriate homomorphisms to Γ.

3. Matching

Let $G = (V, E)$ be a graph. The *matching number* of G, denoted $\nu(G)$, is the size of a maximum matching, and the *deficiency* of G is defined by $\text{def}(G) := |V| - 2\nu(G)$. We let $\text{odd}(G)$ denote the number of components of G that have an odd number of vertices. Note that, for any $X \subseteq V$, we have

$$\text{def}(G) \geq \text{def}(G - X) - |X| \geq \text{odd}(G - X) - |X|;$$

the following theorem shows that equality can be attained.

Theorem 3.1 (Tutte-Berge Formula)**.** *For any graph G,*

$$\text{def}(G) = \max(\text{odd}(G - X) - |X| : X \subseteq V).$$

A set $S \subseteq V$ is *matchable* if there is a matching of G that covers every vertex in S (matchable sets need not have even cardinality). It is well-known that the matchable sets of G form the independent sets of a matroid on V; this is the *matching matroid* of G.

We require some elementary matroid theory. Let M be a matroid with ground set V and let $u, v \in V$. Then, u is a *coloop* of M if u is in every basis of M. The elements u and v are in *series* if neither u nor v are coloops, but there is no basis that avoids both u and v. It is easy to show that series pairs are transitive. That is, if u is in series with v and v is in series with w $(u \neq w)$, then u is in series with w.

Lemma 3.2 (Gallai's Lemma)**.** *If $G = (V, E)$ is a connected graph and $\nu(G - v) = \nu(G)$ for each vertex $v \in V$, then $\mathrm{def}(G) = 1$ and $|V|$ is odd.*

Proof. The matching matroid M of G has no coloops since $\nu(G - v) = \nu(G)$ for each vertex $v \in V$. For each edge uv of G, we have $\nu(G - u - v) < \nu(G)$; that is, u is in series with v. Then, since G is connected, each pair of vertices is in series. Thus, no basis of M can avoid two or more vertices. Therefore, $\mathrm{def}(G) = 1$ and, hence, $|V|$ is odd. □

Proof of the Tutte-Berge Formula. We have already seen that $\mathrm{def}(G) \geq \mathrm{odd}(G - X) - |X|$ for any set $X \subseteq V$, thus it suffices to prove that equality can be attained.

Choose $X \subseteq V$ maximal such that $\nu(G) = \nu(G - X) + |X|$. By our choice of X we have $\nu((G - X) - v) = \nu(G - X)$ for each $v \in V - X$. Then, applying Gallai's Lemma to each component H of $G - X$, we see that $\mathrm{def}(H) = 1$ and $|V(H)|$ is odd. Thus, $\mathrm{def}(G - X) = \mathrm{odd}(G - X)$. Therefore, $\mathrm{def}(G) = |V| - 2\nu(G) = |V| - 2(\nu(G - X) + |X|) = (|V - X| - 2\nu(G - X)) - |X| = \mathrm{def}(G - X) - |X| = \mathrm{odd}(G - X) - |X|$; as required. □

4. A matroid from non-zero A-paths

Throughout this section we let Γ be a group, $G = (V, E)$ be an oriented graph with edge-labels $(\gamma_e : e \in E)$ from Γ, and $A \subseteq V$. We let $\mathrm{def}(G, A, \gamma) := |A| - 2\nu(G, A, \gamma)$.

A *path* is a sequence $P = (v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ where $v_0, \ldots, v_k$ are distinct vertices of G and e_i has ends v_{i-1} and v_i for each $i \in \{1, \ldots, k\}$. Thus, P is ordered in that it has distinguished *start* (v_0) and *end* (v_k). However, P need not be "directed" in that an edge e_i of P may have v_{i-1} or v_i as its head. The path $(v_k, e_k, v_{k-1}, \ldots, v_1, e_1, v_0)$ is denoted by $\bar{P}$. Also, for any $i, j \in \{1, \ldots, k\}$ with $i \leq j$, the path $(v_i, e_{i+1}, v_{i+1}, \ldots, e_j, v_j)$ is denoted by $P[v_i, v_j]$. We allow paths consisting of a single vertex; we refer to such paths as *trivial.*

An *A-collection* is a set $\mathcal{P}$ of vertex disjoint paths such that:

1. each vertex in A is either the start or the end of a path in $\mathcal{P}$,
2. the start of each path $P \in \mathcal{P}$ is in A, and

3. if $P \in \mathcal{P}$ is non-trivial and has its end in A, then $\gamma(P) \neq 0$.

A path $P \in \mathcal{P}$ is *loose* if it is trivial or its end is not in A; thus each path in $\mathcal{P}$ is either an A-path or it is loose (not both). An A-collection is *optimal* if it contains $\nu(G, A, \gamma)$ A-paths; note that there are optimal A-collections.

Let $\Gamma' = \{\gamma(P) \,:\, P \text{ a path of } G\}$ (when Γ is finite we could just take $\Gamma' = \Gamma$). Now, let $S = \{(v, 0) \,:\, v \in A\} \cup \{(v, \gamma) \,:\, v \in V - A,\ \gamma \in \Gamma'\}$. We will define a matroid on the ground set S. Let $\mathcal{P}$ be an A-collection. We let $B(\mathcal{P})$ denote the set of pairs $(v, \gamma(P))$ where v is the end of a loose path $P \in \mathcal{P}$. Note that, $B(\mathcal{P}) \subseteq S$. Now let $\mathcal{B}$ denote the set of all $B(\mathcal{P})$ where $\mathcal{P}$ is an optimal A-collection.

Note that $|B| = \operatorname{def}(G, A, \gamma)$ for all $B \in \mathcal{B}$. Below we show that $\mathcal{B}$ is the collection of bases of a matroid on S. (In the special case that our original A-path problem was just matching, this matroid is isomorphic to the dual of the matching matroid.)

Lemma 4.1. *$\mathcal{B}$ is the set of bases of a matroid on S.*

Proof. As noted above, $\mathcal{B}$ is nonempty and all sets in $\mathcal{B}$ have the same cardinality. Suppose that $\mathcal{B}$ is not the collection of bases of a matroid. Thus, there exist $\mathcal{P}$, $\mathcal{P}'$, and (u, α) satisfying:

4.1.1. *$\mathcal{P}$ and $\mathcal{P}'$ are optimal A-collections and $(u, \alpha) \in B(\mathcal{P}) - B(\mathcal{P}')$ such that for each $(v, \beta) \in B(\mathcal{P}') - B(\mathcal{P})$ we have $(B(\mathcal{P}) - \{(u, \alpha)\}) \cup \{(v, \beta)\} \notin \mathcal{B}$.*

Now:

4.1.2. *we choose $\mathcal{P}$, $\mathcal{P}'$, and (u, α) satisfying 4.1.1 with $|E(\mathcal{P}) - E(\mathcal{P}')|$ as small as possible.*

We use the following claim repeatedly.

4.1.3. *There does not exist an optimal A-collection $\mathcal{P}''$ such that $B(\mathcal{P}) - B(\mathcal{P}'') = \{(u, \alpha)\}$ and $|E(\mathcal{P}'') - E(\mathcal{P}')| < |E(\mathcal{P}) - E(\mathcal{P}')|$.*

Subproof. Suppose that there does exist such an A–collection $\mathcal{P}''$. Since $|B(\mathcal{P}'')| = |B(\mathcal{P})|$ there is a unique element, say (u', α'), in $B(\mathcal{P}'') - B(\mathcal{P})$. Moreover, by 4.1.1, $(u', \alpha') \notin B(\mathcal{P}')$. However, $|E(\mathcal{P}'') - E(\mathcal{P}')| < |E(\mathcal{P}) - E(\mathcal{P}')|$. So, by 4.1.2, $\mathcal{P}''$, $\mathcal{P}$, and (u', α') do not satisfy 4.1.1. That is, there exists an element $(v, \beta) \in B(\mathcal{P}') - B(\mathcal{P}'')$ such that $(B(\mathcal{P}') - \{(u', \alpha')\}) \cup \{(v, \beta)\} \in \mathcal{B}$. However, $(B(\mathcal{P}) - \{(u, \alpha)\}) \cup \{(v, \beta)\} = (B(\mathcal{P}') - \{(u', \alpha')\}) \cup \{(v, \beta)\} \in \mathcal{B}$, contradicting 4.1.1. □

Let $P = (v_0, e_1, v_1, \ldots, e_k, v_k)$ be the path in $\mathcal{P}$ ending at u; thus, $u = v_k$. By possibly reversing the order, we may assume that there is a path $P' = (v'_0, e'_1, v'_1, \ldots, e'_l, v'_l)$ in $\mathcal{P}'$ that starts at v_0. Suppose that P is not contained in P' and let e_a be the first edge of P not in P'. Now let $\mathcal{P}''$ be the A-collection obtained from $\mathcal{P}$ by replacing P with $P[v_0, v_{a-1}]$. Note that, $\mathcal{P}''$ is optimal. Moreover, $B(\mathcal{P}) - B(\mathcal{P}'') = \{(u, \alpha)\}$ and $|E(\mathcal{P}'') - E(\mathcal{P}')| \leq |E(\mathcal{P}) - E(\mathcal{P}')|$; contradicting 4.1.3. Hence, P is contained in P'.

Suppose that P' is disjoint from each path in $\mathcal{P}$ other than the path P, and let $\mathcal{P}''$ be obtained from $\mathcal{P}$ by replacing P with P'. Since $\mathcal{P}$ is optimal, $\mathcal{P}''$ is also optimal and P' is loose. Note that, $(v'_l, \gamma(P')) \in B(\mathcal{P}') - B(\mathcal{P})$ and $(B(\mathcal{P}) - \{(u, \alpha)\}) \cup \{(v'_l, \gamma(P'))\} = B(\mathcal{P}'') \in \mathcal{B}$, contradicting 4.1.1. Therefore, there is some vertex that is both on P' and on a path in $\mathcal{P}$ other than P; let v'_i be the first such vertex on P' and let $Q = (u_0, f_1, u_1, \ldots, f_m, u_m)$ be the path of $\mathcal{P}$ containing v'_i. Suppose that $u_j = v'_i$. We consider two cases.

Case 1: *Q is a loose path.*

Let P_1 be the A-path contained in $P' \cup Q$ and let P_2 be the path in $P' \cup Q$ that starts at u and ends at u_m. Since $\mathcal{P}$ is optimal, $\gamma(P_1) = 0$. Thus, $\gamma(P'[v'_0, v'_i]) = \gamma(Q[u_0, u_j])$ and, hence, $\gamma(P_2) = \gamma(Q)$. Now, let $\mathcal{P}''$ be the A-collection obtained from $\mathcal{P}$ by replacing P and Q with P_2 and the trivial path (u_0). Note that, $B(\mathcal{P}) - B(\mathcal{P}'') = \{(u, \alpha)\}$. Moreover, since $\gamma(P_1) = 0$, $P_1 \neq P'$. Thus, there is an edge of $Q[u_0, u_j]$ that is not in $E(\mathcal{P}')$. So, $|E(\mathcal{P}'') - E(\mathcal{P}')| < |E(\mathcal{P}) - E(\mathcal{P}')|$; contrading 4.1.3.

Case 2: *Q is an A-path.*

Let P_1 and P_2 be the A-paths in $P' \cup Q$ that both start at u and that end with u_0 and u_m respectively. Note that $\gamma(P_1) + \gamma(Q) + \gamma(\bar{P}_2) = 0$ and $\gamma(Q) \neq 0$, so either $\gamma(P_1) \neq 0$ or $\gamma(P_2) \neq 0$. Moreover, either P' is loose (and hence different from P_1 and P_2) or $\gamma(P') \neq 0$. Thus, either $\gamma(P_1) \neq 0$ and $P_2 \neq P'$ or $\gamma(P_2) \neq 0$ and $P_1 \neq P'$. By possibly swapping P_1 and P_2 and reversing Q, we may assume that $\gamma(P_2) \neq 0$ and $P_1 \neq P'$. Let $\mathcal{P}''$ be the A-collection obtained from $\mathcal{P}$ by replacing P and Q with P_2 and the trivial path (u_0). Note that, $B(\mathcal{P}) - B(\mathcal{P}'') = \{(u, \alpha)\}$. Moreover, since $P_1 \neq P'$ there is an edge of $Q[u_0, u_j]$ that is not in $E(\mathcal{P}'') \cup E(P')$. Thus, $|E(\mathcal{P}'') - E(\mathcal{P}')| < |E(\mathcal{P}) - E(\mathcal{P}')|$; contradicting 4.1.3. This final contradiction completes the proof. □

5. Proofs of the main results

Let Γ be a group, $G = (V, E)$ be an oriented graph with edge-labels $(\gamma_e : e \in E)$ from Γ, and $A \subseteq V$. The triple (G, A, γ) is *critical* if

(i) G is connected,
(ii) $\nu(G - \{v\}, A - \{v\}, \gamma) = \nu(G, A, \gamma)$ for each $v \in V$,

(iii) for each $v \in V - A$ and for any edge-labeling γ' obtained from γ by shifting at v we have $\nu(G, A \cup \{v\}, \gamma') > \nu(G, A, \gamma)$, and
(iv) $E(A, \gamma) = \emptyset$.

In Section 3 we defined "coloops" and "series pairs"; in this section we require the dual notions, "loops" and "parallel pairs". Let M be a matroid with ground set S and let $u, v \in S$. Then, u is a *loop* of M if u is not in any basis of M. The elements u and v are *parallel* if neither u nor v are loops, but there is no basis that contains both u and v. Parallel pairs are transitive; that is, if u is parallel with v and v is parallel with w ($u \neq w$), then u is parallel with w.

Lemma 5.1. *Let Γ be a group, let $G = (V, E)$ be an oriented graph with edge labels $(\gamma_e : e \in E)$ from Γ, and let $A \subseteq X$. If (G, A, γ) is critical, then* $\mathrm{def}(G, A, \gamma) = 1$ *and, hence, $|A|$ is odd.*

Proof. Suppose that (G, A, γ) is critical, and let $M = (S, \mathcal{B})$ be the matroid obtained from (G, A, γ) via Lemma 4.1. Let S' denote the set of all non-loop elements of M.

5.1.1. *Let e be and edge of G with tail u and head v, and let $(u, \alpha), (v, \beta) \in S'$. If $\alpha + \gamma_e - \beta \neq 0$, then (u, α) and (v, β) are parallel.*

Subproof. If (u, α) and (v, β) are not parallel, then there is a basis of M that contains them both. That is, there is an optimal A-collection $\mathcal{P}$ with $(u, \alpha), (v, \beta) \in B(\mathcal{P})$. Now, let P_u and P_v be the paths in $\mathcal{P}$ containing u and v respectively. Note that, $P = (P_u, e, \bar{P}_v)$ is an A-path with $\gamma(P) = \alpha + \gamma_e - \beta$. Then, since $\mathcal{P}$ is optimal, we have $\alpha + \gamma_e - \beta = 0$, as required. □

5.1.2. *For each $v \in A$, we have $(v, 0) \in S'$.*

Subproof. Since (G, A, γ) is critical, $\nu(G - v, A - v, \gamma) = \nu(G, A, \gamma)$. Thus, there exists a set $\mathcal{P}$ of $\nu(G, A, \gamma)$ non-zero A-paths each disjoint from v. Now, adding trivial A-paths to $\mathcal{P}$ we obtain an optimal A-collection $\mathcal{P}'$ with $(v, 0) \in B(\mathcal{P}')$. Thus, $(v, 0) \in S'$, as required. □

5.1.3. *For each $v \in V - A$, there exist two distinct elements $(v, \alpha), (v, \beta) \in S'$.*

Subproof. Consider any element $\delta \in \Gamma$, and let γ' be the edge labels obtained from γ by shifting at v by δ. Since (G, A, γ) is critical, $\nu(G, A \cup \{v\}, \gamma') = \nu(G, A, \gamma) + 1$. Let $\mathcal{P}$ be an optimal $A \cup \{v\}$-collection with respect to γ'. Since $\nu(G, A \cup \{v\}, \gamma') > \nu(G, A, \gamma)$, v is the start or end of an $A \cup \{v\}$-path P in $\mathcal{P}$; by possibly reversing P we may assume that v is the end. Then, $\mathcal{P}$ is an optimal A-path collection in G and $\gamma'(P) = \gamma(P) + \delta \neq 0$. Now, $(v, \gamma(P)) \in S'$ and

$\gamma(P) \neq -\delta$. Since δ is any element of Γ, there must exist two distinct elements $(v,\alpha),(v,\beta) \in S'$. □

5.1.4. *Let e be an edge with tail u and head v. Then, there exist $(u,\alpha_u),(v,\alpha_v) \in S'$ that are parallel in M.*

Subproof. First suppose that $u,v \in A$. Let $\alpha_v = \alpha_u = 0$. Since (G,A,γ) is critical, $0 \neq \gamma_e = \alpha_u + \gamma_e - \alpha_v$. Then, by 5.1.1, (u,α_u) and (v,α_v) are parallel. Now we may assume that $u \notin A$ or $v \notin A$; by symmetry we may assume that $v \notin A$. Now, by 5.1.2 and 5.1.3, there exists $\alpha_u \in \Gamma$ such that $(u,\alpha_u) \in S'$, and, by 5.1.3, there exists $\alpha_v \in \Gamma$ such that $(v,\alpha_v) \in S'$ and $\alpha_v \neq \alpha_u + \gamma_e$. Then, by 5.1.1, (u,α_u) and (v,α_v) are parallel. □

For each $v \in V$, let $S'_v = \{(u,\alpha) \in S' : u = v\}$. Consider an optimal A-collection $\mathcal{P}$. Since there is at most one path in $\mathcal{P}$ that ends at v, $|B(\mathcal{P}) \cap S'_v| \leq 1$. Thus, any two elements of S'_v are in parallel. Then, by 5.1.4 and since G is connected, each pair of elements in S' are parallel. Thus, if $\mathcal{P}$ is an optimal A-collection, then $|B(\mathcal{P})| = 1$ and, hence, $\mathrm{def}(G,A,\gamma) = 1$, as required. □

Proof of Theorem 1.2. Choose $X \subseteq V$ maximal such that $\nu(G-X, A-X, \gamma) = \nu(G,A,\gamma) - |X|$. Now among all sets $A' \subseteq V - X$ with $A - X \subseteq A'$ and edge-labelings γ' obtained from γ by shifting on the vertices in $A' - A$ such that $\nu(G-X, A', \gamma') = \nu(G-X, A-X, \gamma)$ we choose the pair (A', γ') with A' as large as possible. Now let $H_1, \dots, H_l$ be the components of $G - X - E(A', \gamma')$ and let A'_i denote $A' \cap V(H_i)$. Note that,

$$\nu(G,A,\gamma') = |X| + \sum_{i=1}^{l} \nu(H_i, A'_i, \gamma').$$

By our choice of X and A', it is easy to check that each of the triples (H_i, A'_i, γ') is critical. Then, by Lemma 5.1, $\nu(H_i, A'_i, \gamma') = \left\lfloor \frac{|A'_i|}{2} \right\rfloor$. So,

$$\nu(G,A,\gamma') = |X| + \sum_{i=1}^{l} \left\lfloor \frac{|A'_i|}{2} \right\rfloor,$$

as required. □

Proof of Theorem 1.1. By Theorem 1.2 there exist edge-labels $(\gamma'_e : e \in E)$ obtained by shifting γ at vertices in $V - A$ and there exist sets $X, A' \subseteq V$ such that $A - X \subseteq A' \subseteq V - X$ and

$$\nu(G,A,\gamma) = |X| + \sum_{i=1}^{l} \left\lfloor \frac{|V(H_i) \cap A'|}{2} \right\rfloor,$$

where $H_1, \ldots, H_l$ are the components of $G - X - E(A', \gamma')$. Let $i \in \{1, \ldots, l\}$ and let A'_i denote $V(H_i) \cap A'$. Now, let $X_i \subseteq A'_i$ with $|X_i| = |A'_i| - 1$, and let $X^* = X \cup X_1 \cup \cdots \cup X_l$. Note that, $\nu(H_i - X_i, A'_i - X_i, \gamma') = 0$ since $|A'_i - X_i| = 1$. Now,

$$\begin{aligned}\nu(G - X^*, A - X^*, \gamma) &= \nu(G - X^*, A - X^*, \gamma') \\ &\leq \nu(G - X^*, A' - X^*, \gamma') \\ &= \nu(G - X^* - E(A', \gamma'), A' - X^*, \gamma') \\ &\leq \sum_{i=1}^{l} \nu(H_i - X_i, A'_i - X_i, \gamma') \\ &= 0.\end{aligned}$$

Thus, X^* meets every non-zero A-path in G. Suppose that $\nu(G, A, \gamma) < k$. Then,

$$\begin{aligned}2k - 2 &\geq 2\nu(G, A, \gamma) \\ &= 2|X| + \sum_{i=1}^{l} 2 \left\lfloor \frac{|A'_i|}{2} \right\rfloor \\ &\geq |X| + \sum_{i=1}^{l} (|A_i| - 1) \\ &= |X| + \sum_{i=1}^{l} |X_i| \\ &= |X^*|,\end{aligned}$$

as required. □

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