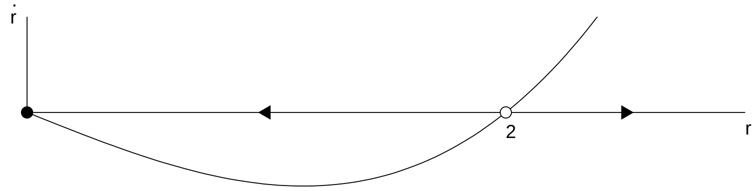
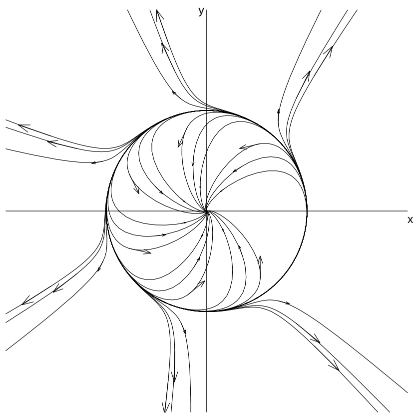
Solutions 4

7.1.1 (Page 228)

The radial and angular dynamics are uncoupled and so can be analyzed seperately. Treating $\dot{r} = r^3 - 4r$ as a vector field on the line, we see that $r^* = 0$ is a stable fixed point and $r^* = 2$ is unstable.

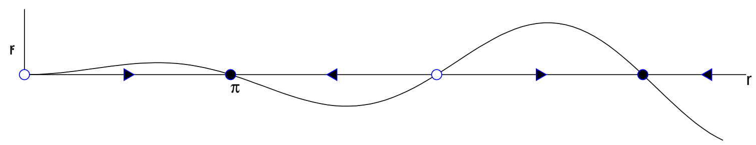


Hence, back in the phase plane, all trajectories inside the circle $r^* = 2$ approach the origin monotonically. All trajectories outside this circle approach the infinity. Since the motion in the θ-direction is simply rotation at constant angular velocity, we see that all trajectories spiral asymptotically away from a limit circle at $r = 2$.

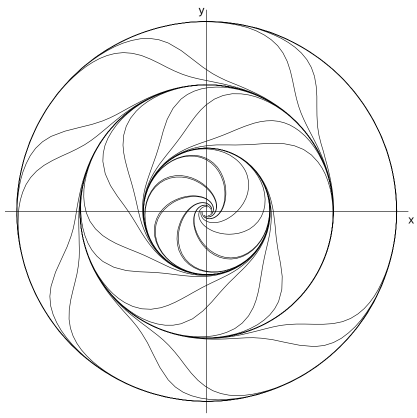


7.1.4 (Page 228)

Again, the radial and angular dynamics are uncoupled and so can be analyzed seperately. Treating $\dot{r} = r\sin(r)$ as a vector field on the line, we see that $r^* = (2n-1)\pi$ are stable fixed points and $r^* = 2n\pi$ are unstable, where n is a positive integer.



Hence, back in the phase plane, since the motion in the θ-direction is simply rotation at constant angular velocity, all trajectories between the circles $r^* = 2n\pi$ and $r^* = 2(n+1)\pi$ spiral asymptotically towards the limit circles $r = (2n-1)\pi$.



7.2.6 (Page 229)

This is an example of the procedure above to find the potential function V.

a) First notice that the system satisfies

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x},$$

so we can find a $V$by integrating f and g. V should look like

$$V(x,y) = -\int_0^x (y^2 + y\cos x)\mathrm{dx} + a(y) = -xy^2 - y\sin x + a(y),$$

where $a(y)$ satisfies

$$g(x,y) = -\frac{\partial V}{\partial y} = 2xy + \sin x - a'(y).$$

We can simply choose $a(y) \equiv 0$, and obtain the potential function

$$V(x,y) = -xy^2 - y\sin x$$

.

b) Again we can find a $V$by integrating f and g. V should look like

$$V(x,y) = -\int_0^x (3x^2 - 1 - e^{2y})\mathrm{dx} + a(y) = -x^3 + x + xe^{2y} + a(y),$$

where $a(y)$ satisfies

$$g(x,y) = -\frac{\partial V}{\partial y} = -2xe^{2y} - a'(y).$$

We can simply choose $a(y) \equiv 0$, and obtain the potential function

$$V(x,y) = -x^3 + x + xe^{2y}$$

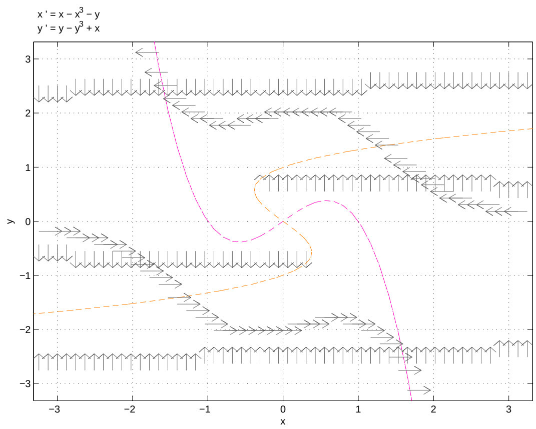
.

7.2.10 (Page 230)

Consider $V(x,y) = ax^2 + by^2$. Then $\dot{V} = 2ax\dot{x} + 2by\dot{y} = 2ax(y - x^3) - 2by(x + y^3) = 2(a-b)xy - 2(ax^4 + by^4)$. If we choose $a = b > 0$, then $V > 0$ and $\dot{V} < 0$ for all $(x,y) \neq (0,0)$. Hence $V = ax^2 + by^2$ is a Liapunov function and so there are not closed orbits.

7.3.3 (Page 231)

First we plot the nullclines



We can find the trapping zone by doing the following. We start with point (a,a), where $a > 0$. It sits on the y-nullcline if $a + a - a^3 = 0$ or $a = \sqrt{2}$, and at this point, $\dot{x} < 0$. By symmetry, the point $(-\sqrt{2}, \sqrt{2})$ is sitting on the x-nullcline and $\dot{y} < 0$. We use the same argument for the point $(-\sqrt{2}, -\sqrt{2})$ and $(\sqrt{2}, -\sqrt{2})$ and combine with the figure above we find that the squre which four corners are sitting on the points $(\pm\sqrt{2}, \pm\sqrt{2})$ is the trapping zone.

Now since the Jacobian $A = \begin{pmatrix} 1 - 3x^2 & -1 \\ 1 & 1 - 3y^2 \end{pmatrix}$ evaluates to $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ at the origin. The eigenvalues are $1 \pm i$ so it is an unstable spiral. Therefore the repelling fixed point drives all neighbouring trajectories into the trapping region and we can apply the Poincaré-Bendixson theorem here to show that there is a periodic solution.

7.3.4 (Page 231)

a) The Jacobian $A = \begin{pmatrix} 1-12x^2-y^2\frac{y}{2} & -2xy-\frac{1+x}{2} \\ -8xy-2-4x & 1-4x^2-3y^2 \end{pmatrix}$ evaluates to $\begin{pmatrix} 1 & -\frac{1}{2} \\ -2 & 1 \end{pmatrix}$ at the origin. The eigenvalues are $0, 2$ so it is an unstable fixed point.

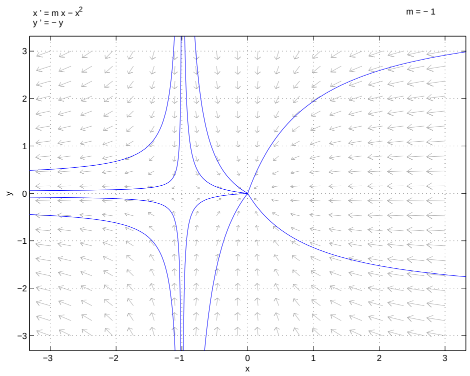
b) Let $V = (1-4x^2-y^2)^2$, then

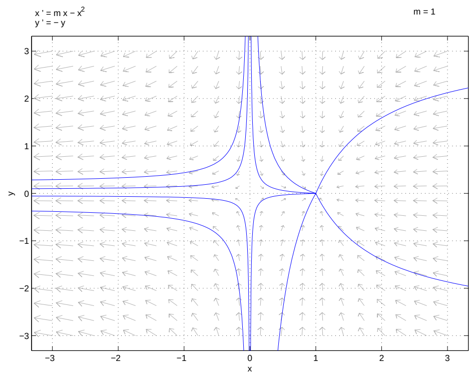
$$\begin{aligned}
\dot{V} &= -4(1-4x^2-y^2)(4x\dot{x}+y\dot{y}) \\
&= -4(1-4x^2-y^2)\{4x[x(1-4x^2-y^2)-\frac{y(1+x)}{2}]+y[x(1-4x^2-y^2)+2x(1+x)]\} \\
&= -4(1-4x^2-y^2)^2(4x^2+y^2)
\end{aligned}$$

Since $\dot{V} < 0$ unless $V = 0$ or $x = y = 0$. Since $(0, 0)$ is an repeller, all trajectories will approach the ellipse $4x^2 + y^2 = 1$ as $t \to \infty$.

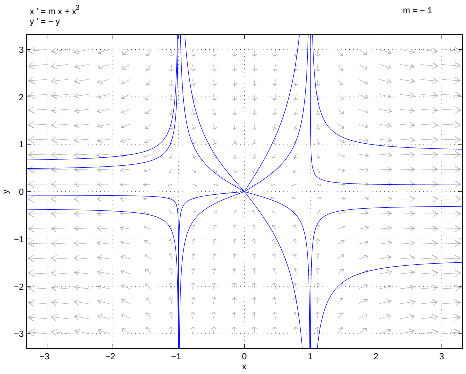
8.1.1 (Page 284)

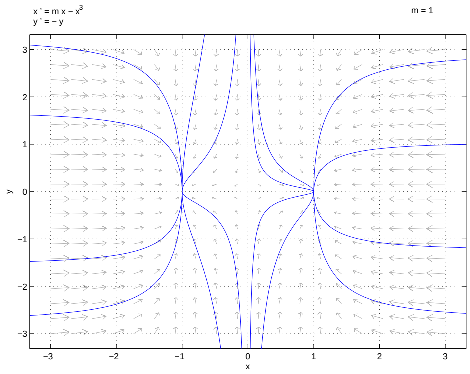
a) The stable fixed points are $(0, 0)$ for $\mu \leq 0$ and $(\mu, 0)$ for $\mu > 0$. The eigenvalues are $-|\mu|, -1$.





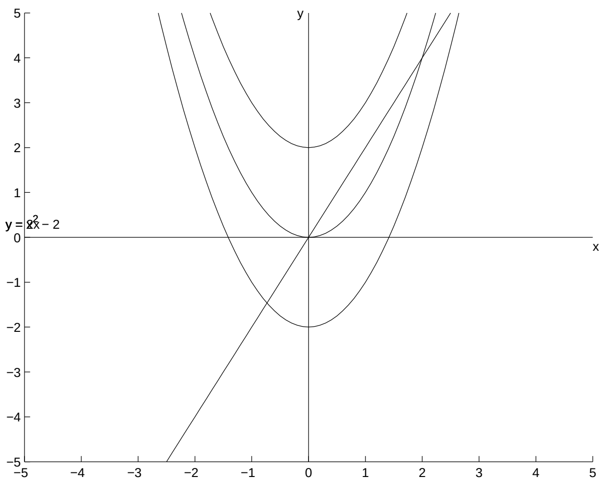
b) The stable fixed points are $(0, 0)$ for $\mu \leq 0$ and $(\pm\sqrt{\mu}, 0)$ for $\mu > 0$. The eigenvalues are also $\mu, -1$ for $\mu \leq 0$ and $-2\mu, -1$ for $\mu > 0$.





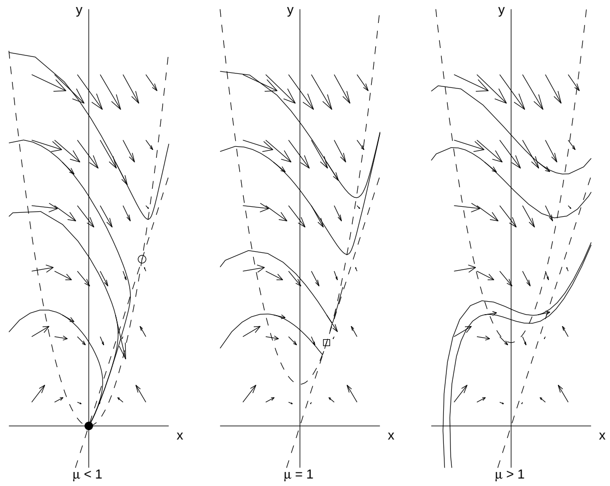
8.1.6 (Page 285)

a) The nullclines are drawn for $\mu = -2$, 0, and 2.



b) The fixed points are the intersections of the nullclines. From the above picture it's clear that as μ increases, two fixed points first collide then disappear. The collision occurs when the parabola $y = x^2 + \mu$ is tangential to the line $y = 2x$. Since the slope of the tangent line of the parabola at point $(t, t^2 + \mu)$ is $2t$, if the tangent line is $y = 2x$, then $t = 1$. Therefore $y = x^2 + \mu$ is tangential to $y = 2x$ at point $(1, 1 + \mu)$, which implies that bifurcation occurs at $\mu = 1$. The motion of the fixed points described above also shows that this is a saddle-node bifurcation.

c)

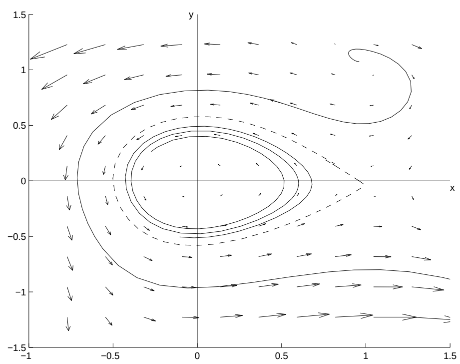


8.2.2 (Page 287)

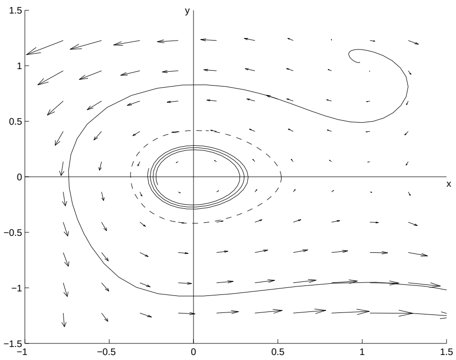
The Jacobian $A = \begin{pmatrix} \mu + y^2 & -1 + 2xy \\ 1 - 2x & \mu \end{pmatrix}$ evaluates to $\begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$ at the origin. If $\mu = 0$, we have $\tau = 0$ and $\Delta = 1$, then the eigenvalues satisfying $\lambda^2 + 1 = 0$ are $\pm i$, i.e. they are pure imaginary.

8.2.3 (Page 287)

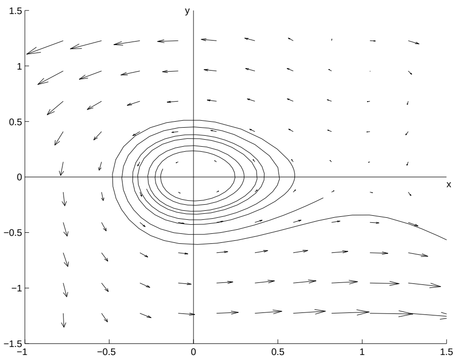
We start with a negative μ very close to 0. The figure below shows the phase portrait for $\mu = -0.04$, there is an unstable cycle around the origin:



As μ increases to -0.02, the cycle gets closer to the origin:



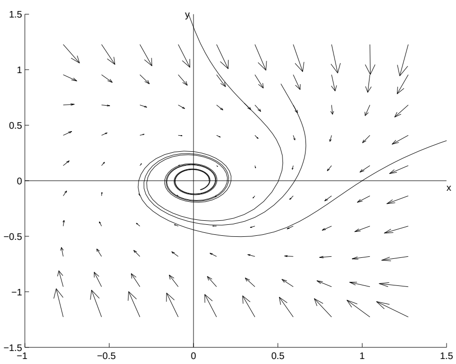
Now if μ becomes positive, the unstable limit cycle disappears and the origin becomes unstable:



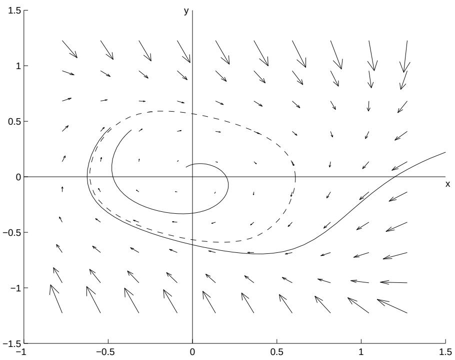
Therefore our computer experiments suggest that this Hopf bifurcation is subcritical.

8.2.6 (Page 287)

When μ is negative, say $\mu = -0.04$, the phase portrait looks like



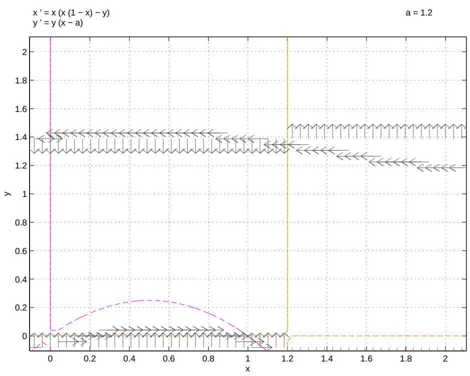
The origin is a stable spiral. Now let $\mu = 0.4$ be a small positive number, then a stable limit cycle appears, and the origin becomes an unstable spiral:

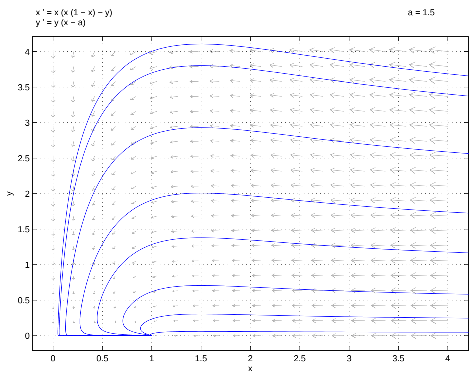


This experiment suggests that the bifurcation is supercritical.

8.2.8 (Page 287)

a,c) Here we show the nullclines in the first quadrant (left) and the phase portrait for $a = 1.2$ (right). We can see that all the predators go extinct ($y \to 0$).



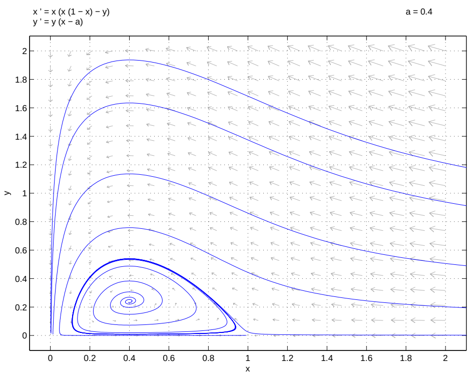


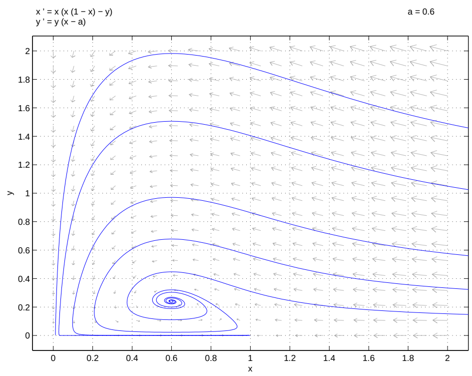
b) The Jacobian $A = \begin{pmatrix} 2x - 3x^2 - y & -x \\ y & x - a \end{pmatrix}$ evaluates to $\begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix}$ for the origin. The eigenvalues are $0, -a$ and it is isolated fixed point. A evaluates to $\begin{pmatrix} -1 & -1 \\ 0 & 1-a \end{pmatrix}$ for $(1, 0)$. The eigenvalues are $-1, 1-a$. It is a stable node for $a > 1$ and a saddle for $a < 1$. A evaluates to $\begin{pmatrix} a - 2a^2 & -a \\ a - a^2 & 0 \end{pmatrix}$ for $(a, a - a^2)$. The eigenvalues are $\frac{(1-2a\pm\sqrt{4a^2-3})a}{2}$ and it is a unstable spiral when $0 \le a < \frac{1}{2}$, a stable spiral when $\frac{1}{2} < a < 1$ and a saddle when $a > 1$.

d) It is a supercritical Hopf bifurcation at $a_c = \frac{1}{2}$, when the real parts of the eigenvalues simultaneously cross the imaginary axis into the right half plane.

e) Near $a_c = \frac{1}{2}$, the frequency of the limited cycle oscillation is about $\frac{\sqrt{3-4a_c^2}}{2} = \frac{1}{\sqrt{2}}$.

f) Here we show the phase portraits for $a = 0.4$ (left) and for $a = 0.6$ (right).





8.2.12 (Page 289)

a) For this system, $f(x,y) = xy^2, g(x,y) = -x^2, \omega = 1$. To find a, note that the only derivatives term that is non-zero for $(x,y) = (0,0)$ is $f_{xyy} = 2$ and $g_{xx} = -2$. Thus, $a = \frac{1}{8}$.

b) From (a), the Hopf bifurcation occurs is subcritical, same as what we found in 8.2.3.

8.4.2 (Page 291) The radial and angular dynamics are uncoupled and so can be analyzed seperately. Treating $\dot{r} = r(\mu - sin(r))$ as a vector field on the line, we see that for $|\mu| > 1, r^* = 0$ is the only fixed point and it is stable for $\mu < -1$ and unstable for $\mu > 1$. At $\mu = -1$, the system undergoes a saddle node bifurcation of cylcles at $r^* = (2n + \frac{3}{2})\pi$. For $-1 < \mu < 0, r^* = 0, (2n-1)\pi - \arcsin\mu, 2n\pi + \arcsin\mu$. The stabilities of fixed points alternate with increasing r^*, starting with $r^* = 0$ being stable. At $\mu = 0$ the zero fixed point undergoes a supercritical Hopf bifurcation and a new fixed point $r^* = \arcsin\mu$ emerges. For $0 < \mu < 1, r^* = 0, \arcsin\mu, (2n-1)\pi - \arcsin\mu, 2n\pi + \arcsin\mu$. The stabilities of fixed points alternate with increasing r^*, starting with $r^* = 0$ being unstable. Subsequently, the system undergoes another saddle node bifurcation of cycles when $\mu = 1$ in which pairs of adjacent fixed points $[\arcsin\mu, \pi - \arcsin\mu], [2\pi + \arcsin\mu, 3\pi - \arcsin\mu] \ldots$ collide at the $\frac{\pi}{2}, 2\pi + \frac{\pi}{2}, \ldots$. Since the motion in the θ-direction issimply rotation at constant angular velocity, we see that all trajectories spiral asymptotically to or from limit circles at $r = r^*$.