Solutions 3

5.1.10 (Page 141)

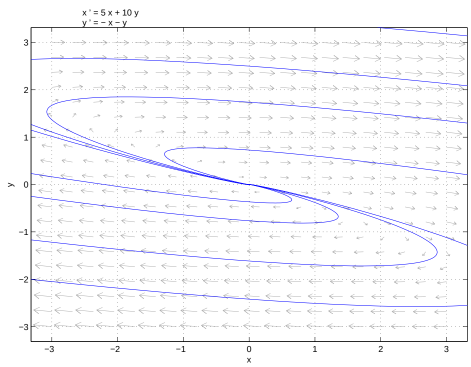
a) The coefficient matrix $\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$ has $\tau = 0$, so the origin is a center and it is Liapunov stable.

c) In this system x is never changed. If we start at a point (x_0, y_0) where x_0 is positive, then the solution of $\dot{y} = x_0$ is $y = x_0 t$, which goes arbitrarily large in the long run. So in this case the origin does not have any type of stability.

e) Notice that the system is already decoupled. Since 0 is a stable fixed point for both systems of x and y, an arbitrary flow starting at any point always gets closer to the origin as time evolves, so it is asymptotically stable.

5.2.4 (Page 143)

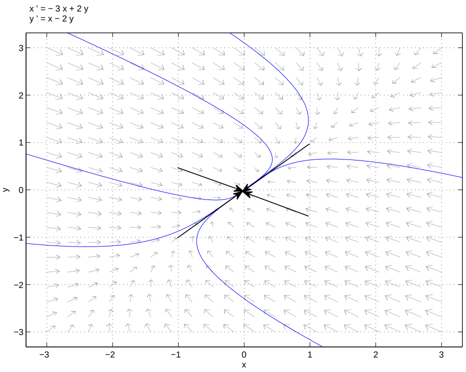
The coefficients matrix is $\begin{pmatrix} 5 & 10 \\ -1 & -1 \end{pmatrix}$, which has $\tau = 4$, $\Delta = 5$. The characteristic equation is $\lambda^2 - 4\lambda + 5 = 0$, hence $\lambda_1 = 2 + \mathrm{i}, \lambda_2 = 2 - \mathrm{i}$.. The corresponding eigenvectors are $\mathbf{v}_1 = \binom{-3-\mathrm{i}}{1}, \mathbf{v}_2 = \binom{-3+\mathrm{i}}{1}$.



The origin is an unstable spiral.

5.2.6 (Page 143)

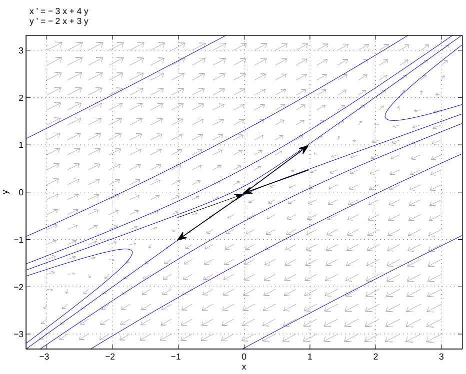
The coefficients matrix is $\begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}$, which has $\tau = -5$, $\Delta = 4$. The characteristic equation is $\lambda^2 + 5\lambda + 4 = 0$, hence $\lambda_1 = -4, \lambda_2 = -1$. The corresponding eigenvectors are $\mathbf{v}_1 = \binom{-2}{1}, \mathbf{v}_2 = \binom{1}{1}$.



The origin is a stable node. The arrows indicate the two eigenvectors.

5.2.8 (Page 143)

The coefficients matrix is $\begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix}$, which has $\tau = 0$, $\Delta = -1$. The characteristic equation is $\lambda^2 - 1 = 0$, hence $\lambda_1 = 1, \quad \lambda_2 = -1$. The corresponding eigenvectors are $\mathbf{v}_1 = \binom{1}{1}, \mathbf{v}_2 = \binom{2}{1}$.

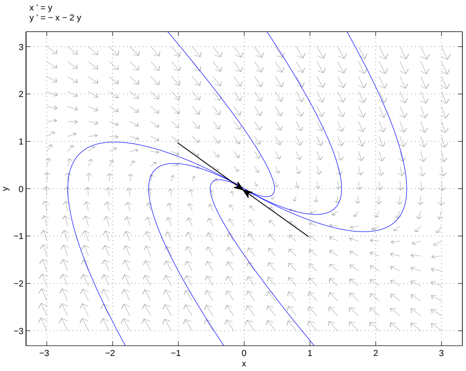


The origin is a saddle point. The arrows indicate the two eigenvectors.

5.2.10 (Page 143)

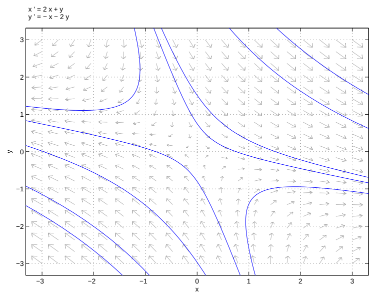
The coefficients matrix is $\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$, which has $\tau = -2$, $\Delta = 1$. The characteristic equation is $\lambda^2 + 2\lambda + 1 = 0$, hence $\lambda_1 = \lambda_2 = -1$. An eigenvector $\mathbf{v} = (v_1, v_2)$ satisfies

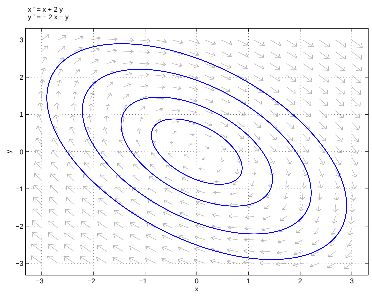
$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \binom{v_1}{v_2} = \binom{0}{0}$, which has a nontrivial solution $(v_1, v_2) = (-1, 1)$. Since there is only one eigenvector, the origin is a degenerate node.



5.3.4 (Page 144)

The coefficients matrix is $\begin{pmatrix} a & b \\ -b & -a \end{pmatrix}$, which has $\tau = 0$, $\Delta = -a^2 + b^2$. The characteristic equation is $\lambda^2 - a^2 + b^2 = 0$, hence $\lambda_1 = \sqrt{a^2 - b^2}, \lambda_2 = -\sqrt{a^2 - b^2}$. The corresponding eigenvectors are $\mathbf{v}_1 = \binom{1}{\frac{\sqrt{a^2-b^2}-a}{b}}, \mathbf{v}_2 = \binom{0}{-\frac{\sqrt{a^2-b^2}+a}{b}}$. If $a^2 - b^2 > 0$ then the origin is a saddle point and the relationship will be explosive. Their feelings are opposite, since $\frac{sqrta^2-b^2-a}{b} < 0$ (see the left figure for $a = 2, b = 1$). If $a^2 - b^2 < 0$ then the origin is a centre and the relationship will be cyclical. The (see the right figure for $a = 1, b = 2$).



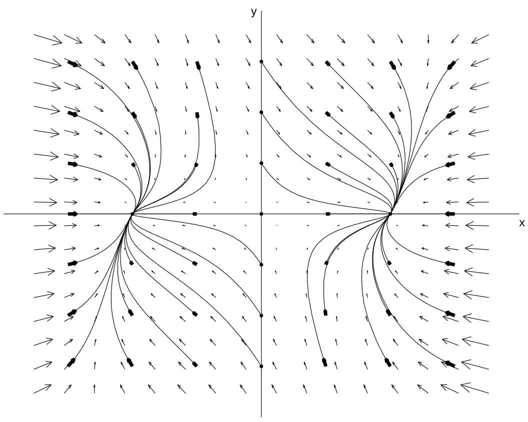


6.3.4 (Page 182)

First we find the fixed points by solving $\dot{x} = 0$, $\dot{y} = 0$ simultaneously. Hence we need $x = 0$ or $x = \pm 1$, and $y = 0$. The Jacobian matrix at a general point (x, y) is

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 - 3x^2 & 1 \\ 0 & -1 \end{pmatrix}.$$

Next we evaluate A at the fixed points. At $(0, 0)$, we find $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, so $(0, 0)$ is a saddle point. At $(\pm 1, 0)$, $A = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$, so both $(1, 0)$ and $(-1, 0)$ are attractors.



6.3.9 (Page 183)

a) By solving $\dot{x} = 0$, $\dot{y} = 0$ simultaneously. we get $x = y = 0$ or $x = y = \pm 2$. The Jacobian matrix at a general point (x, y) is

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} -4 & 3y^2 \\ -3 & 3y^2 - 1 \end{pmatrix}.$$

Next we evaluate A at the fixed points. At $(0, 0)$, we find $A = \begin{pmatrix} -4 & 0 \\ -3 & -1 \end{pmatrix}$, so $(0, 0)$ is an attractor. At $(\pm 2, \pm 2)$, $A = \begin{pmatrix} -4 & 12 \\ -3 & 11 \end{pmatrix}$, so both $(2, 2)$ and $(-2, -2)$ are saddles.

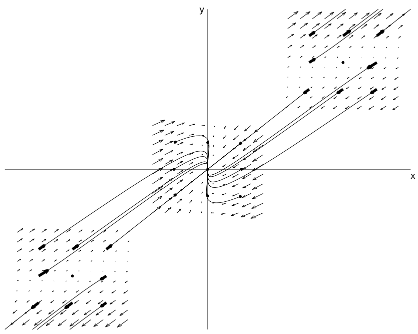
b) Let $u = x - y$, then

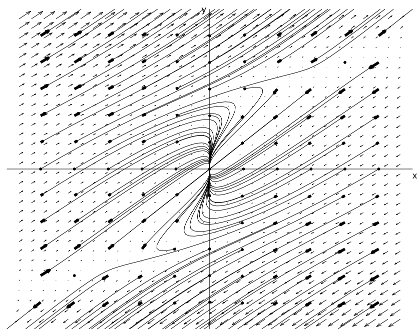
$$\dot{u} = \dot{x} - \dot{y} = (y^3 - 4x) - (y^3 - y - 3x) = y - x = -u.$$

Any trajectory that starts on $x = y$ also starts on $u = 0$. But $u = 0$ is a fixed point of $\dot{u} = -u$, so the trajectory will stay at $u = 0$, i.e. $x = y$.

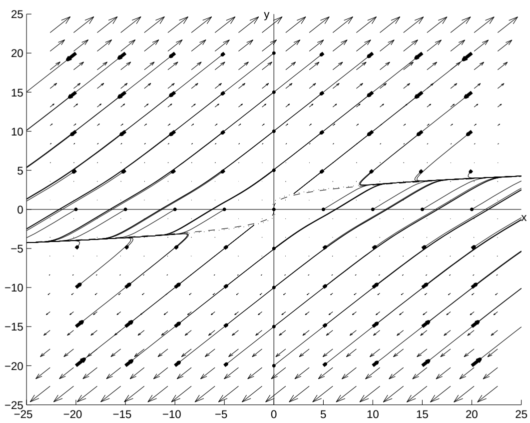
c) It can also be seen from $\dot{u} = -u$ that $u = 0$ is a stable fixed point. So $|u(t)| \to 0$ as $t \to \infty$.

d) To "sketch" the phase portrait, one should start locally at the fixed points $(0, 0)$ and $(\pm 2, \pm 2)$, then connect some flows to get a better idea what the phase portrait looks like. At this stage the picture should look like the first one below. Then one tries to fill the picture by guessing and get the second picture.





e) In part d) I used domain $-3 \le x, y \le 3$. In the bigger domain $-20 \le x, y \le 20$ the trajectories seem to approach a curve as $t \to \infty$:



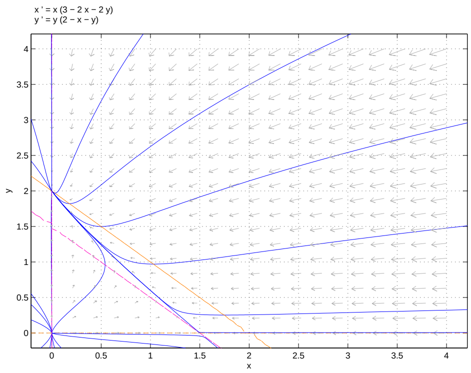
In the above picture two curves are drawn: $y^3 - y = 3x$ and $y^3 = 3x$. You can only see one curve (the dashed one) there because they are barely distinguishable. You can use either one as an approximation of the real curve. These two curves are good because as $t \to -\infty$, most trajectories will lie between them and they're very close. To prove the trajectories are below the first curve, one can use the fact that it is a nullcline. To prove they are above the second curve, one can look at the derivative of $y^3 - 3x$.

6.4.3 (Page 184)

The fixed points of the system are $(x^*, y^*) = (0,0), (0,2)$ and $(\frac{3}{2}, 0)$. We compute the Jacobian:

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 - 4x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}.$$

Next we evaluate A at the fixed points. At $(0,0)$, we find $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, so $(0,0)$ is unstable node. At $(0,2)$, $A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$, so $(0,2)$ is a stable node. At $(\frac{3}{2}, 0)$, $A = \begin{pmatrix} -3 & 3 \\ 0 & \frac{1}{2} \end{pmatrix}$, so $(\frac{3}{2}, 0)$ is a saddle point. The basin of attraction is $x > 0, y \geq 0$.



6.5.1 (Page 185)

a) This can be rewritten as the vector field

$$\dot{x} = y$$
$$\dot{y} = x^3 - x$$

where y represents the particle's velocity. Equilibrium points occur where $(\dot{x}, \dot{y}) = (0,0)$. Hence the equilibria are $(x^*, y^*) = (0,0)$ and $(\pm 1, 0)$. To classify fixed points we compute the Jacobian:

$$A = \begin{pmatrix} 0 & 1 \\ 3x^2 - 1 & 0 \end{pmatrix}.$$

At $(0,0)$, we have $\tau = 0$, $\Delta = 1$, so the origin is a center. But when $(x^*, y^*) = (\pm 1, 0)$, we find $\Delta = -2$; hence these equilibria are saddles.

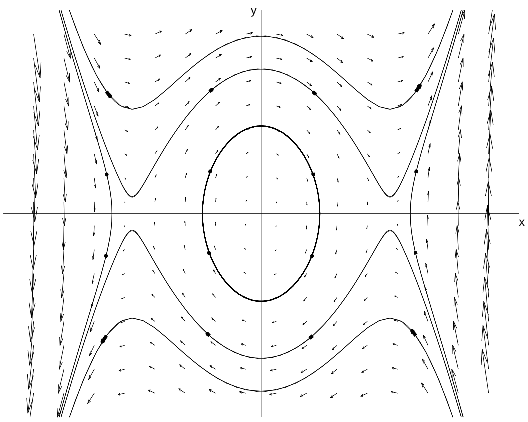
b) Any antiderivative of $x - x^3$ induces a conserved quantity, for example

$$V(x) = \frac{1}{2}x^2 - \frac{1}{4}x^4$$

gives us a conserved quantity

$$E(x) = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4.$$

c) Since this system is conservative, the point we predicted to be a center by the linear approximation should indeed be a center. The phase portrait looks as follows:



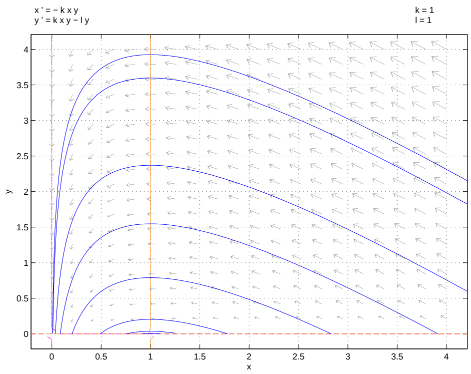
6.5.6 (Page 185)

a) The fixed points are $(x^*, y^*) = (\hat{x}, 0), \hat{x} > 0$. The Jacobian matrix at a general point (x, y) is

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} -ky & -kx \\ ky & kx - l \end{pmatrix}.$$

Next we evaluate A at the fixed points. At $(\hat{x}, 0)$, we find $A = \begin{pmatrix} 0 & -k\hat{x} \\ 0 & k\hat{x} - l \end{pmatrix}$, so $(\hat{x}, 0)$ are non-isolated fixed points.

b,d) The dashed lines indicate the nulliclines. As $t \to \infty$, $y(t) \to 0$ and $x(t) \to x_\infty \in [0, \frac{l}{k})$.

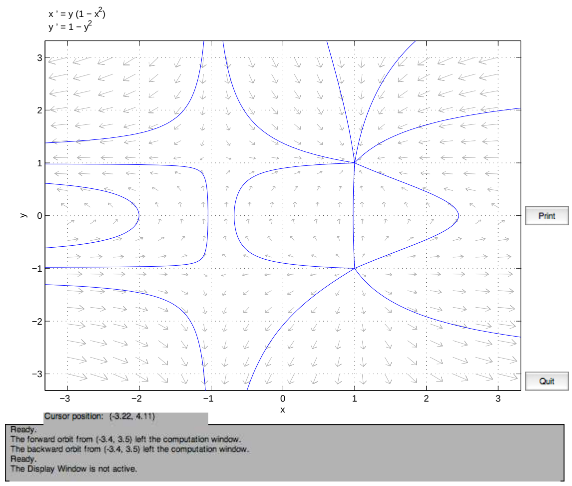


c) Note that $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{kxy - ly}{-kxy} = -1 + \frac{l}{kx}$. Separate the variables and integrate on both sides, we have $g(x, y) = y + x - \frac{l}{k} \log x$ as a conserved quantity for the system.

e) We can see from the phase portrait that an epidemic occurs when $x_0 > \frac{l}{k}$.

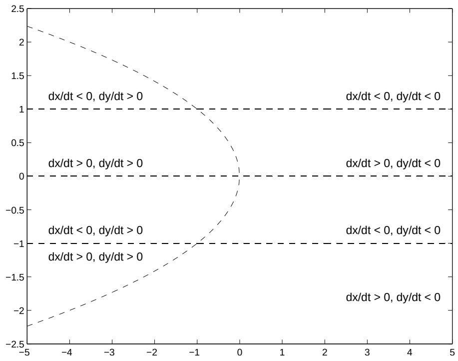
6.6.1 (Page 190)

The system is invariant under the change of variables $t \to -t$ and $y \to -y$. Hence the system is reversible.



6.6.6 (Page 185)

a, b) For $\dot{x}$, it is positive when $0 < y < 1$ or $y < -1$ and negative when $-1 < y < 0$ or $y > 1$. For $\dot{y}$, it is positive when $x < -y^2$ and negative when $x > -y^2$ (see the figure below, where the dashed lines are nullclines).



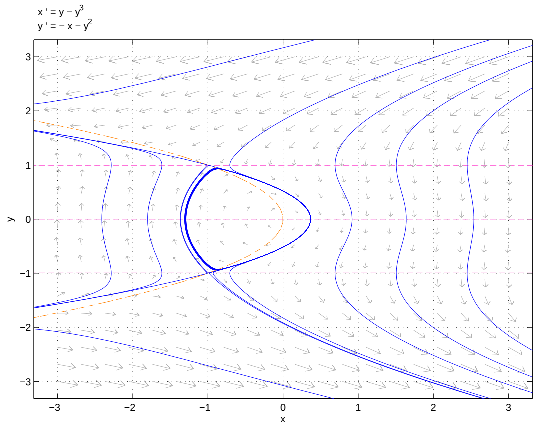
c) The Jacobian is

$$A = \begin{pmatrix} 0 & 1-3y^2 \\ -1 & -2y \end{pmatrix}.$$

At $(-1, \pm 1)$, we have $\tau = \mp 2, \Delta = -2$. The eigenvalues are $\lambda_1 = \mp 1 + \sqrt{3}$ and $\lambda_2 = \mp 1 - \sqrt{3}$. The corresponding eigenvectors are $\mathbf{v}_1 = \binom{1}{\pm\frac{1}{2} - \frac{\sqrt{3}}{2}}, \mathbf{v}_2 = \binom{1}{\pm\frac{1}{2} + \frac{\sqrt{3}}{2}}$.

d) Consider the unstable manifold of the saddle point $(-1, -1)$. This manifold leaves the saddle point along the vector $\mathbf{v} = \binom{-1}{\frac{1}{2} + \frac{\sqrt{3}}{2}}$. Hence part of the unstable manifold lies in the region where $\dot{x} < 0, \dot{y} > 0$. A phase point along the manifold will move up and to the left and eventually will cut the negative x-axis. Since the system is reversible under the transformation $t \to -t, y \to -y$. By reversibility, there must be a twin trajectory with the same endpoint on the negative x-axis from $(1, 1)$ but with arrow reversed. Together the two trajectories form a heteroclinic orbit.

e) Consider the unstable manifold of the saddle point $(-1, 1)$. This manifold leaves the saddle point along the vector $\mathbf{v} = \binom{1}{\frac{1}{2} - \frac{\sqrt{3}}{2}}$. Hence part of the unstable manifold lies in the region where $\dot{x} > 0, \dot{y} < 0$ (Note that $\frac{1}{2} - \frac{\sqrt{3}}{2} > -2$). A phase point along the manifold will move down and to the right and eventually will cut the positive x-axis (the origin is a centre). Since the system is reversible under the transformation $t \to -t, y \to -y$. By reversibility, there must be a twin trajectory with the same endpoint on the positive x-axis from $(1, -1)$ but with arrow reversed. Together the two trajectories form another heteroclinic orbit.



x ' = y − y^3
y ' = − x − y^2
x
y