Note Title 03/12/2007

Fall 2005 - Midterm 2

$\boxed{P4}$ $\begin{cases} \dot{x} = a(1-2b)x + y - ax(x^2+y^2) \\ \dot{y} = -x + ay - ay(x^2+y^2) \end{cases}$

(a) -(h) $|ab| < 1$, $b < 1/2$

(a) linear analysis of the f.p. $(0,0)$

$$A\big|_{(0,0)} = \begin{bmatrix} a(1-2b) & 1 \\ -1 & a \end{bmatrix}$$

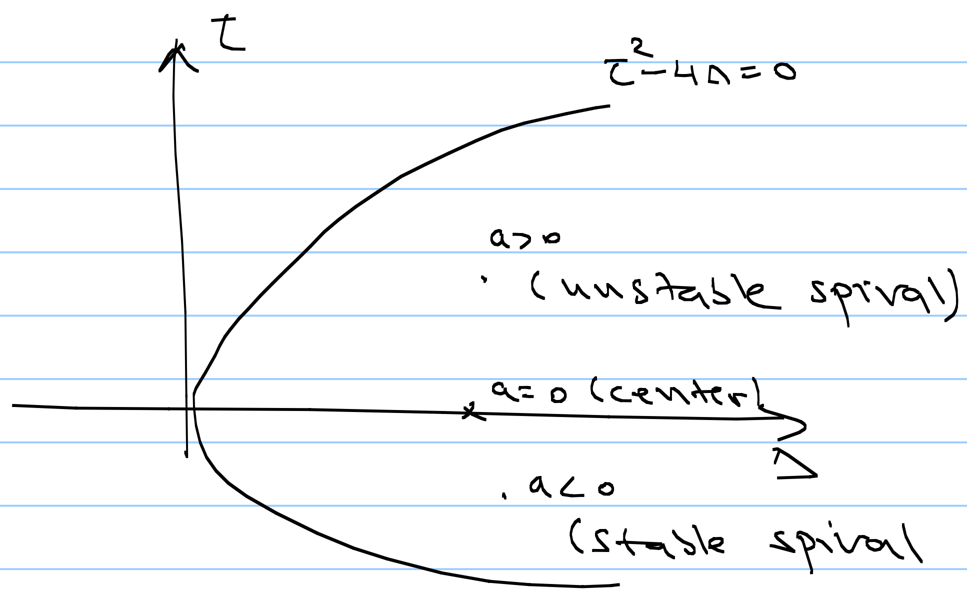
$$\tau = a(1-2b) + a = 2a\underbrace{(1-b)}_{>0} \quad (b < 1/2 < 1)$$

$$\text{sign}\,\tau \sim \text{sign}\,a$$

$$\Delta = a^2\underbrace{(1-2b)}_{>0\ (b<1/2)} + 1 > 0$$

$$\tau^2 - 4\Delta = 4a^2(1-b)^2 - 4a^2(1-2b) - 4 =$$

$$= 4a^2(\cancel{1} - \cancel{2b} + b^2 - \cancel{1} + \cancel{2b}) - 4 = 4(\underbrace{a^2b^2 - 1}_{<0\ (|ab|<1)})$$



b) Convert to polar coordinates

$$r\dot{r} = x\dot{x} + y\dot{y} \quad , \qquad \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$$

$$r\dot{r} = a(\underline{\underline{1}}-2b)x^2 + \cancel{xy} - ax^2(x^2+y^2) - \cancel{xy} + \underline{\underline{ay^2}} - ay^2(x^2+y^2)$$

$$= a(x^2+y^2) - 2abx^2 - a(x^2+y^2)(x^2+y^2)$$

$$= ar^2 - 2ab\,r^2\cos^2\theta - a\,r^2 \cdot r^2$$

$$\Rightarrow \quad \dot{r} = ar - ar^3 - 2abr\cos^2\theta$$

$$= ar(1-r^2) - 2abr\cos^2\theta$$

$$\dot{\theta} = \frac{1}{r^2}\Big(x\left(-x + ay - ay(x^2+y^2)\right) - y\big(a(1-2b)x + y - \\ - ax(x^2+y^2)\big)\Big)$$

$$= \frac{1}{r^2}\Big(-x^2 + \cancel{axy} - \cancel{axy(x^2+y^2)} - \cancel{axy} + 2abxy - y^2 + \\ + \cancel{axy(x^2+y^2)}\Big)$$

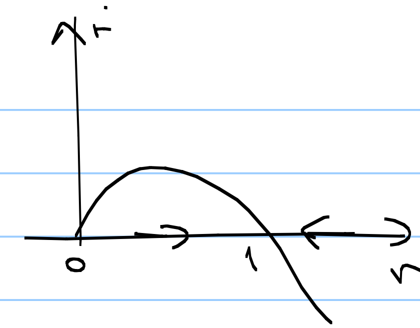
$$= \frac{1}{r^2}\left(-r^2 + 2abr^2\cos\theta\sin\theta\right)$$

$$= -1 + 2ab\cos\theta\sin\theta$$

So:
$$\begin{cases} \dot{r} = ar(1-r^2) - 2abr\cos^2\theta \\ \dot{\theta} = -1 + 2ab\sin\theta\cos\theta \end{cases}$$

(c) Take $b=0$, show that there is exactly one limit cycle for $a>0$

$b=0$: $\begin{cases} \dot r = ar(1-r^2) \\ \dot\theta = -1 \end{cases}$



$r=1$ attracting

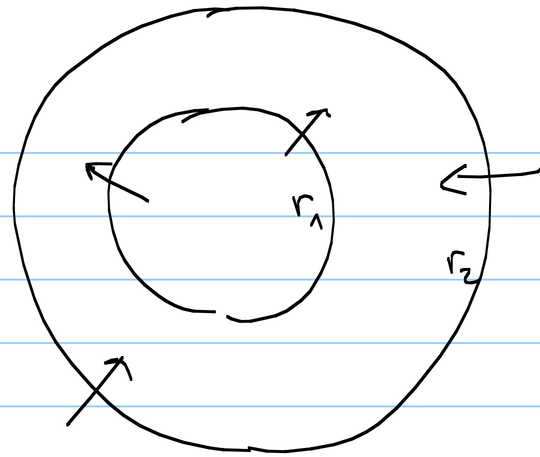
$\dot\theta = -1$ rotation (clockwise)

(d) – (g) $a>0$, $b<1/2$, $|ab|<1$

(d) Find a circle of max radius r_1 s.t. $\dot r \geq 0$ along $r=r_1$

(e) — " — min — " — r_2 $\dot r \leq 0$ along $r=r_2$

(f) Prove that there is at least one limit cycle for this range of parameters



$\dot r = ar(1-r^2) - 2abr\cos^2\theta.$

Two cases: $b>0$, $b<0$

$\boxed{b>0}$

$$ar(1-r) - 2abr \le ar(1-r^2) - \underbrace{2abr\cos^2\theta}_{\geq 0} \le ar(1-r^2)$$

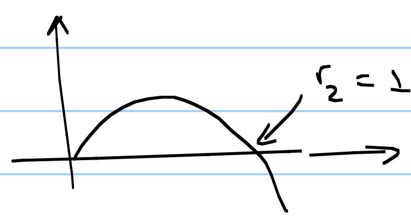
with equality for $\cos^2\theta = 1$

with eq. for $\cos\theta = 0$

r_1 – the largest r for which $ar(1-r^2) - 2abr \geq 0$

r_2 – the smallest r for which $ar(1-r^2) \le 0$

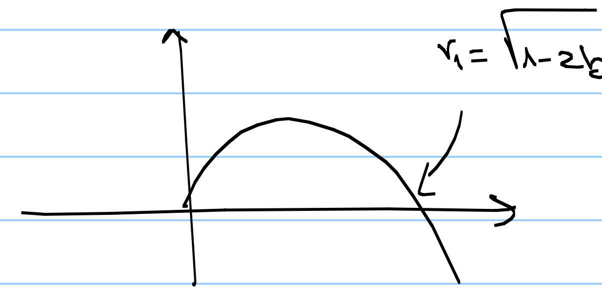
r_2:



$ar(1-r^2) \le 0$ for $r \geq 1$

r_1: $\quad ar(1-r^2) - 2abr = ar\left(1-2b-r^2\right)$

recall $\quad b < 1/2 \Rightarrow 1-2b > 0$



$ar\left(1-2b-r^2\right) \geq 0 \quad$ for $\quad r \leq \sqrt{1-2b}$

Therefore $\quad \boxed{r_1 = \sqrt{1-2b}, \quad r_2 = 1} \quad$ if $\quad b > 0$

Note $\quad r_1 < r_2$

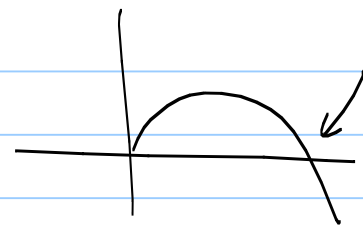
Case $b<0$ $\qquad \dot{r} = ar(1-r^2) \underbrace{-2abr\cos\theta}_{>0}$

$ar(1-r^2) \leq ar(1-r^2) - 2abr\cos\theta \leq ar(1-r^2) - 2abr$

Now: $\quad r_1$ – largest r for which $\quad ar(1-r^2) \geq 0$

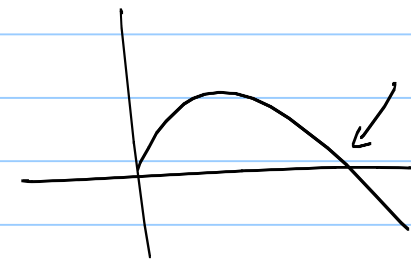
r_2 – smallest r for which $ar(1-r^2) - 2abr \le 0$

r_1: $r_1 = 1$



r_2: $ar(1-2b-r^2) \le 0$

$r_2 = \sqrt{1-2b}$



again, $r_1 < r_2$

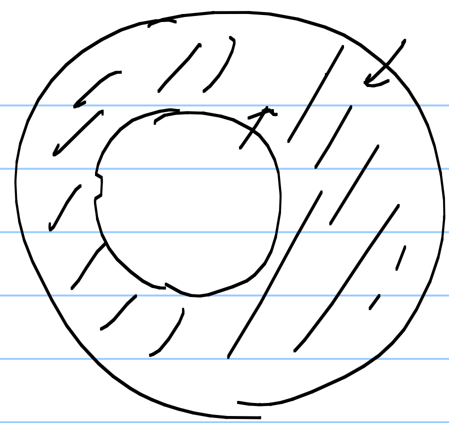
$r_1 = r_2$ for $b = 0$

Summary: $b > 0$ $r_1 = \sqrt{1-2b}$, $r_2 = 1$

$(a > 0)$ $b < 0$ $r_1 = 1$, $r_2 = \sqrt{1-2b}$

(7) Existence of a limit cycle – Poincaré – Bendixson.

– trapping region $r_1 \le r \le r_2$

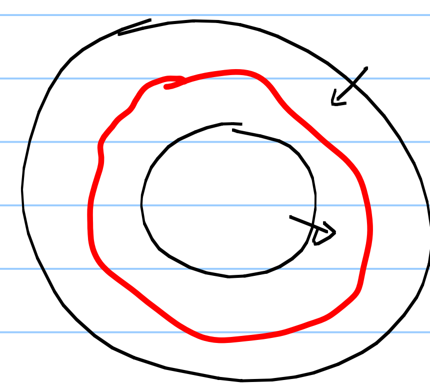


We also have to show that there are no f.p in this region!

(not hard to see that)

(g) If there is exactly one limit cycle, determine its stability. Show that if there are several limit cycles, they all have the same period depending on a and b.

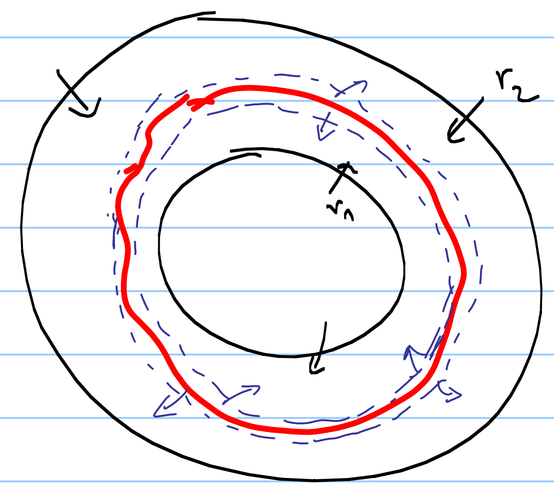
first question:



limit cycle - - unique !

or just P–B (a traj that stays in R ↗ closed orbit ↘ spiral toward a c.o.)

Idea: make an argument that: if the limit cycle was unstable, it would repel trajectories in a small "tube" around it. That would imply existence of at least two other



limit cycles. Contradiction

Conclusion: The l.c has to be stable

second question (period) $\dot{\theta} = -1 + 2ab\cos\theta\sin\theta$

For a closed trajectory $\theta = \theta_0 \longrightarrow \theta = \theta_0 + 2\pi$

(can choose $\theta_0 = 0$)

$$\frac{d\theta}{-1+2ab\cos\theta\sin\theta} = dt \Rightarrow \int_0^{2\pi} \frac{d\theta}{-1+2ab\cos\theta\sin\theta} = \int_0^T dt$$

T - the period of the closed traj.

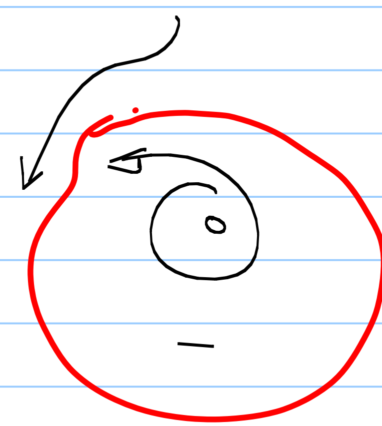
$$\Rightarrow \quad T = \int_0^{2\pi} \frac{d\theta}{-1+2ab\cos\theta\sin\theta}$$ for all closed traj.!

(h) Bifurcation at $a=0$

$a>0$ origin unstable spiral

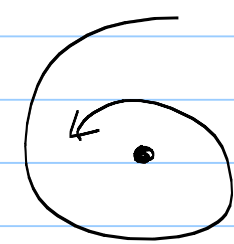
\+ limit cycle

the size of the limit cycle does not depend on a!



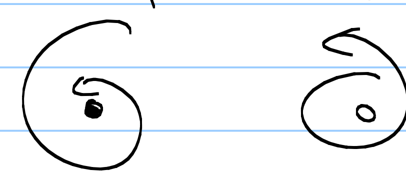
For instance, when $b>0$, the cycle is trapped in the annulus $\sqrt{1-2b} < r < 1$, away from the origin

$a<0$ origin - stable spiral



We expect (needs to be verified) that there exists a l.c for $a<0$ as well, but it is unstable. Or maybe there are no limit cycles at all for $a<0$.

<u>$a=0$</u> degenerate Hopf

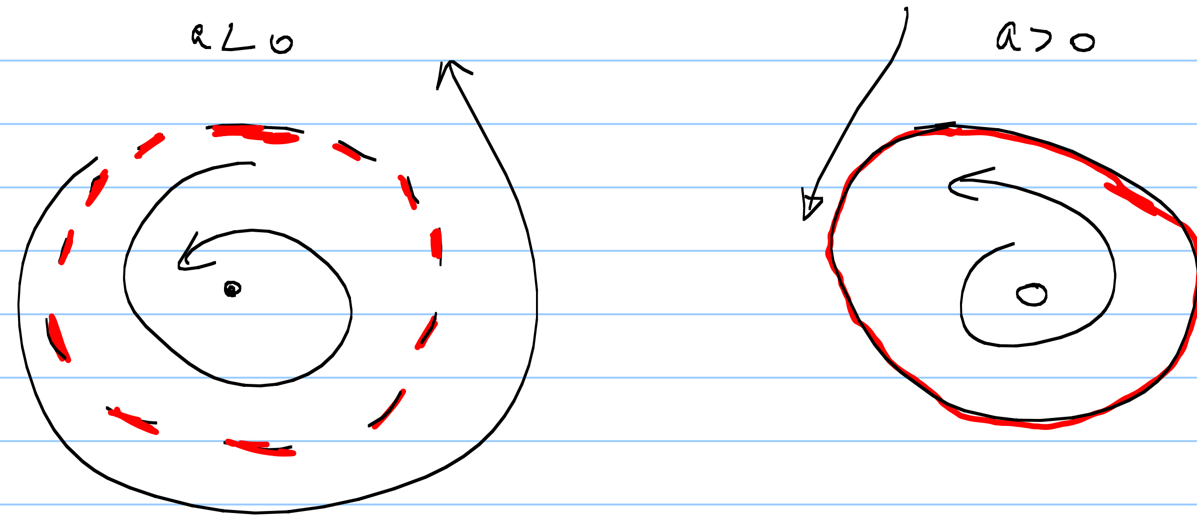


$a<0$ $a>0$

$\dot{r} = ar(1-r^2) - 2abr^2\cos\theta$

$a < 0$ $\qquad a\left[r(1-r^2) - 2br\cos\theta \right]$

Construction of r_1, r_2



(i) Other cases $b > \frac{1}{2}$ etc.

variations: ...