A dispersive regularization of the modulational instability of stratified gravity waves

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Abstract

We derive high-order corrections to a modulation theory for the propagation of internal gravity waves in a density-stratified fluid with coupling to the mean flow. The methodology we use allows for strong modulations of wavenumber and mean flow, extending previous approaches developed for the quasi-monochromatic regime. The wave mean flow modulation equations consist of a system of nonlinear conservation laws that may be hyperbolic, elliptic or of mixed type. We investigate the regularizing properties of the asymptotic correction terms in the case when the system becomes unstable and ill-posed due to a change of type (loss of hyperbolicity). A linear analysis reveals that the regularization by the added correction terms does so by introducing a short-wave cut-off of the unstable wavenumbers. We perform various numerical experiments that confirm the regularizing properties of the correction terms, and show that the growth of unstable modes is tempered by nonlinearity. We also find an excellent agreement between the solution of the corrected modulation system and the modulation variables extracted from the numerical solution of the nonlinear Boussinesq equations.

1 Introduction

The propagation of internal gravity waves represent an important feature of the dynamics of the ocean and the atmosphere. The gravity waves in a stratified fluid are produced by buoyancy disturbances, created when a fluid parcel is vertically displaced to a region with a different density. A significant component of the literature devoted to the study of such wave motions in fluids relates to modulation theory [1], an asymptotic method that describes the time evolution of slowly-varying wavetrains. As an asymptotic analysis, the choice of modulation theory is particularly appropriate for atmospheric gravity waves, as the vertical variations in profiles of density stratification and horizontal wind shear from surface to stratosphere are often significant, yet typically slowly-varying [2, 3].

Early works by Drazin [2] and Rarity [4] investigated the propagation of modulated sinusoidal waves of finite amplitude. These works omitted however a critical feature of the nonlinear regime, which is the coupling of waves with the mean flow. The later works of Bretherton [5, 6] and Grimshaw [3, 7] addressed this issue and developed a modulation theory for the fully nonlinear case. As presented in Grimshaw [3] for instance, the equations governing the dynamics of the modulated variables are a system of nonlinear conservation laws for the wavenumber in the vertical direction, the wave action density and the horizontal mean flow. This system of quasi-linear PDEs may be hyperbolic, elliptic or of mixed type, depending on the solution itself.

In the hyperbolic regime, the modulation system exhibits finite-time breakdown via shock formation. The failure of the modulation theory due to wave breaking is a shortcoming of the method that has been under-addressed in the literature. In our recent work [8], we consider next-order asymptotic corrections to the modulation theory for the linear Boussinesq equations. We find that the correction terms introduce dispersion that significantly alters the steepening process that precedes the shock singularity. The dispersive regularization effects are similar to those seen in the small dispersion limit of the Korteweg-deVries equation [9]. However, [8] presents only the modulation theory for the linear Boussinesq, where no coupling with the mean flow is present.

An equally under-addressed issue is the breakdown of modulated wavetrains when the governing system of conservation laws loses hyperbolicity. In this instance the breakdown is due to to the exponential growth of the solution in regions where the system is elliptic [3] . In his work, Rarity [4] makes a detailed investigation of the elliptic region and suggests the need for introducing jump conditions at the hyperbolic-elliptic boundary, similar to the discontinuities across the boundary of a subsonic bubble in compressible fluid flow. In [4], the elliptic domain is associated with a trapping region, within which waves can no longer propagate. Rarity also draws attention to the fact that the experiments indicate that small amplitude internal waves are able to negotiate the trapping region, without necessarily acquiring large amplitudes or degenerating into quasi-turbulence, while preserving the sense of the direction of propagation.

In the present paper we take the ideas from [8] further and develop a corrected modulation theory for the nonlinear Boussinesq equations, that accounts for weak wave coupling to the mean flow. The quasi-linear system of equations is augmented with high-order, nonlinear correction terms that change, in a fundamental way, the behavior of the solutions. In the hyperbolic case, the shock formation is prevented by the dispersive effects of the asymptotic corrections in a similar manner to what was observed and reported in [8]. The novelty regards the elliptic instability, where we present numerical and analytical evidence that the asymptotic corrections regularize the dynamics of the modulated wavetrain, by suppressing the growth of large wavenumbers.

Related studies were recently done in [10, 11]. Sutherland [10] investigated numerically and analytically a weakly nonlinear, quasi-monochromatic modulation theory and compared the results with fully nonlinear simulations of the Boussinesq equations. The author also discussed modulation instability and the role of third-order dispersion in stabilizing the unstable wavepackets. The study [10] concludes that the modulation equation accurately describes the interaction between the waves and the wave-induced mean flow before the onset of parametric instability. Tabaei and Akylas [11] extended a previous study performed by Shrira [12] and considered a modulation theory for *flat* wavetrains which yields resonant instabilities. Their numerical observations indicate that no elliptic instability is observed provided the modulation equations are augmented with a certain additional term that allows for high-order dispersive effects. The additional term is derived from a higher-order modulation theory performed in the small-amplitude, quasi-monochromatic regime.

The methodology we develop in our paper is more general that those from [10, 11], as it applies to strong (but slowly-varying) modulations. We work within the weakly nonlinear regime in order to derive asymptotically valid correction terms, but numerical experiments indicate, similar to observations from [11], that the same asymptotic corrections can also be used to regularize the dynamics of finite-amplitude wavetrains. Higher-order modulation theories and their possible impact on the dynamics have been addressed since early works. For instance, in his book (Chaper 15.5, [1]), Whitham derives a high-order modulation theory for a nonlinear Klein-Gordon equation and briefly comments on the stabilizing effects of the correction terms. Other discussions of higher-order extensions for modulation theory appear in the early literature [13, 14, 15], but they have found little use in applications, and seldom have their regularization effects have been investigated numerically.

An atmospheric context for the breakdown of waves is the upward transport of momentum to the stratosphere that is an important process in the phenomenon known as the quasi-biennial oscillation (QBO) [16, 17, 18]. The QBO is an atmospheric climate phenomenon where the equatorial winds in the stratosphere reverse in (east/west) direction with a period averaging approximately 28 months [19]. The atmospheric QBO was reproduced in a laboratory experiment by Plumb and McEwan [20], as well as illustrated in various computational (virtual) experiments [21]. In their recent work, Wedi and Smolarkewicz [21] investigated the turbulent breakdown of an inertia gravity wavetrain and find that viscous dissipation and critical level absorption are mechanisms secondary to instabilities from nonlinear flow interactions. It is this line of research that motivates our specific efforts to refine modulation theory applied to atmospheric gravity waves.

In the next section, we highlight the main results of this study. We write down the modulation equations, describe briefly the regularization procedure and discuss the effects that the correction terms have on the dynamics. Section 3 presents the derivation of the corrected modulation theory for the nonlinear Boussinesq equations. In Section 4 we investigate the type of the modulation system and perform a linear stability analysis to get a better account for the effects of the correction terms. Section 5 shows several numerical experiments that illustrate the regularizing properties of the correction terms, as well as the close approximation of the corrected modulation theory when compared against the solution of the full Boussinesq equations.

2 Regularized modulation theory — highlights

In this section we give a brief overview of the corrected modulation theory for the nonlinear Boussinesq equations. The details of the derivation and the numerical results are presented in the subsequent sections.

The nonlinear Boussinesq equations for a 2-dimensional stratified fluid are given by

$$\eta_t + u\eta_x + w\eta_z + b_x = 0 \tag{1a}$$

$$b_t + ub_x + wb_z + N^2 w = 0, \tag{1b}$$

where x and z are, respectively, the horizontal and vertical spatial coordinates, $u(x, z, t)$ and $w(x, z, t)$ represent, respectively, the horizontal and vertical components of the velocity, $\eta(x, z, t) = u_z - w_x$ denotes the vorticity and $b(x, z, t)$ is the buoyancy disturbance. The Brunt-Väisälä frequency N is assumed to be constant; by rescaling we can take $N = 1$.

One of the main concerns of this study is how the horizontal mean flow couples with the wave propagation. To this purpose we decompose the velocities and the buoyancy fields into mean and wave components. The velocities are thus expressed in the form $u = U(z, t) + \tilde{\psi}_z$ and $w = -\tilde{\psi}_x$, where incompressibility is imposed through a disturbance streamfunction $\tilde{\psi}(x, z, t)$.

The horizontal mean flow $U(z,t)$ allows for vertical shear, but as is typical for stratified flow, the vertical velocity is assumed to have no mean flow. Likewise, the buoyancy field is expressed as $b = B(z,t) + \tilde{b}$. The mean components are distinguished from wave components by imposing disturbances to have zero mean. The y-component of vorticity is $\eta = U_z(z,t) + \tilde{\eta}$, where its wave part satisfies $\tilde{\eta} = \nabla^2\tilde{\psi}$. This application of modulation theory requires the assumption that the mean flow $U(z,t)$ and the mean buoyancy $B(z,t)$ are slowly-varying functions of z and t, relative to the wavelength of the wave.

The disturbance equations (1) are now

$$\tilde{\eta}_t + U\tilde{\eta}_x + \tilde{\eta}_x\tilde{\psi}_z - \tilde{\eta}_z\tilde{\psi}_x + \tilde{b}_x = -U_{zt} + U_{zz}\tilde{\psi}_x \tag{2a}$$

$$\tilde{b}_t + U\tilde{b}_x + \tilde{b}_x\tilde{\psi}_z - \tilde{b}_z\tilde{\psi}_x - \tilde{\psi}_x = -B_t + B_z\tilde{\psi}_x \tag{2b}$$

$$\nabla^2\tilde{\psi} - \tilde{\eta} = 0. \tag{2c}$$

We consider a small-amplitude wavetrain solution to (2),

$$\begin{aligned}\begin{pmatrix}\tilde{b}\\ \tilde{\eta}\\ \tilde{\psi}\end{pmatrix} &= \mathrm{Re}\left\{\epsilon\begin{pmatrix}\alpha_1\\ \beta_1\\ \gamma_1\end{pmatrix}e^{i\phi} + \epsilon^3\begin{pmatrix}\alpha_2\\ \beta_2\\ \gamma_2\end{pmatrix}e^{2i\phi} + \text{h.o.t.}\right\}\\ &= \mathrm{Re}\left\{\epsilon\mathbf{A_1}(z,t)e^{i\phi} + \epsilon^3\mathbf{A_2}(z,t)e^{2i\phi} + \text{h.o.t.}\right\},\end{aligned} \tag{3}$$

that is dominated locally by a single plane wave whose phase is $\phi(x,z,t)$. As we focus on possible breakdown in the vertical propagation of waves, the modulation theory is simplified by restricting to exact periodicity in x, and considering only spatial modulations in the z-direction. Thus, the phase gradient $\phi_x = k$ is assumed constant, while the vertical wavenumber $\phi_z(z,t)$ and frequency $-\phi_t(z,t)$ will be slowly-varying. The phase $\phi(x,z,t)$ is taken to be real-valued. The higher-order terms (h.o.t.) include all other harmonics, with the exception that zeroth harmonics are accounted for by mean fields. Finally, the Boussinesq equations (1) have the property that its linearized (single-mode) plane wave solutions are also exact nonlinear solutions by fortuitous cancellation of the quadratic advection terms. As a consequence, harmonic generation by a modulated plane wave is weaker than usual, and this is reflected in the $O(\epsilon^3)$ scaling of the second harmonic in (3).

Modulation theory assumes that the wavetrain properties vary on a long spatial scale $\zeta = \epsilon z$ and evolve on a slow time scale $\tau = \epsilon t$, where ϵ is small [6]. Therefore, the vector amplitudes $\mathbf{A_1}(z,t)$, $\mathbf{A_2}(z,t)$, the phase gradients $\phi_t(z,t)$, $\phi_z(z,t)$, the mean flow $U(z,t)$ and the mean buoyancy $B(z,t)$ are assumed to depend on the slow variables $\zeta = \epsilon z$ and $\tau = \epsilon t$. Choosing the ϵ for the modulation theory to be the same as the ϵ-scaling for the wave amplitude (3) will result in the mean flow coupling having the same order as the dispersive corrections. Also note that the modulation theory developed here allows for *strong* modulations of the vertical wavenumber and extends previous approaches [10, 11] which apply only in the quasi-monochromatic regime.

As derived by Grimshaw [3], the dynamics of the modulated wavetrain can be specified by the time evolution of the wavenumber in the vertical direction $m(\zeta,\tau) \equiv \phi_z$, the wave action $\mathcal{F}(\zeta,\tau)$ and the horizontal mean flow $U(\zeta,\tau)$. The modulation equations coupled with the mean flow are given by the following system of conservation laws:

$$m_\tau + (\Omega + kU)_\zeta = 0 \tag{4a}$$

$$\mathcal{F}_\tau + (C\mathcal{F})_\zeta = 0 \tag{4b}$$

$$U_\tau + \epsilon^2 \frac{k}{2} (C\mathcal{F})_\zeta = 0, \tag{4c}$$

where the plane wave frequency Ω and the group velocity are given by

$$\Omega^2 = \frac{k^2}{k^2 + m^2}, \qquad C = \frac{\partial \Omega}{\partial m} = -\frac{m\Omega^3}{k^2}.$$

The ϵ^2 in the mean flow equation (4c) arises due to the weak wave amplitude assumed in (3). Analyzing the characteristic speeds for this conservation law system (4) shows it to be elliptic in regions where $m^2 < k^2/2$ [10, 11] — a result independent of ϵ and the mean flow coupling. This modulation instability of low vertical wavenumber waves is a mode of failure of the modulation theory which is additional to the wavebreaking by gradient steepening, characteristic to hyperbolic conservation laws, investigated by the authors in [8].

However, in a systematic derivation of the modulation theory, dispersive corrections appear at order $O(\epsilon^2)$. In [8] we show that the higher-order corrections can be introduced by replacing Ω in (4a) with ω given by the $O(\epsilon^2)$ corrected dispersion relation

$$\omega^2 = \frac{k^2}{k^2 + m^2 + \epsilon^2 R}.$$

The added term R is a regularization term that has a fairly complicated expression involving the modulation variables and their (up to second order) spatial derivatives. With this regularization, the time evolution of the modulation system is $O(\epsilon^2)$ accurate. The regularization R introduces third-order dispersive effects to (4a) and fundamentally changes the system dynamics. This paper investigates the interplay between the mean flow coupling and the dispersive corrections.

A linear analysis of the regularized system shows that the instability present at *all* wavenumbers in the elliptic region for (4) has now reduced drastically to a finite band at small wavenumbers. Similar stabilizing effects are produced by high-order terms in the modulation theories of nonlinear Klein-Gordon equation (see Chapter 15.5, [1]), water waves [22] and nonlinear Schrödinger (NLS) equation (see Chapter 2d, [23]). In these systems, the instability (also known as the Benjamin-Feir instability) manifests initially by an exponential growth of the bandwidth, but this growth is eventually bounded by the action of nonlinearities and high-order terms. Likewise, for the high-order modulation theory developed in this work, numerical experiments show bounded solutions that are in excellent agreement with the full Boussinesq equations, even when initial data lies entirely in the elliptic domain $m^2 < k^2/2$ of the unregularized equations (4).

3 Derivation of the modulation equations

The derivation of the corrected modulation equations follows the procedure we developed for the linear Boussinesq equations in [8]. The calculations are more involved in the nonlinear case however, as we have to account for the contribution of the mean flow.

We employ the notations

$$m(\zeta,\tau) \equiv \phi_z \qquad ; \qquad \omega(\zeta,\tau) \equiv -(\phi_t + kU), \tag{5}$$

for the wavenumber in the vertical direction and the reduced frequency.

Equality of the mixed partials in terms of the slow scales gives the conservation law

$$m_\tau + (\omega + kU)_\zeta = 0. \tag{6}$$

Corrected modulation theory. Introduce the perturbation expansion (3) into (2) and set to 0 the coefficient of $e^{i\phi}$. We get [1]

$$[\mathcal{M} + \epsilon\mathcal{L}_1 + \epsilon^2\mathcal{L}_2]\mathbf{A_1} = O(\epsilon^4), \tag{7}$$

where the matrix operators $\mathcal{M}$, $\mathcal{L}_1$, $\mathcal{L}_2$ are given by

$$\mathcal{M} = \begin{pmatrix} k & -\omega & 0 \\ -\omega & 0 & -k \\ 0 & -k & -k(k^2+m^2) \end{pmatrix}, \tag{8}$$

$$\mathcal{L}_1 = i\begin{pmatrix} 0 & -\partial_\tau & 0 \\ -\partial_\tau & 0 & 0 \\ 0 & 0 & k(2m\partial_\zeta + m_\zeta) \end{pmatrix}, \tag{9}$$

$$\mathcal{L}_2 = \begin{pmatrix} 0 & 0 & -kU_{\zeta\zeta} \\ 0 & 0 & 0 \\ 0 & 0 & k\partial_\zeta^2 \end{pmatrix}. \tag{10}$$

Note that equation (2c) has been multiplied by ik to give a symmetric leading-order matrix $\mathcal{M}$.

We start by inspecting (7). At leading-order, the zero determinant condition $\det(\mathcal{M}) = 0$ yields the dispersion relation

$$\omega^2 = \frac{k^2}{k^2+m^2} \equiv \Omega(m)^2. \tag{11}$$

We use $\Omega(m)$ to denote the dispersion relation (11), to distinguish from the corrected frequency, ω, which has the former as the leading-order. We likewise identify the group velocity

$$c = c(m,\omega) = \frac{\partial\omega}{\partial m} = -\frac{m\omega^3}{k^2} \tag{12}$$

as distinguished from its leading-order

$$C = c(m,\Omega) = -\frac{m\Omega^3}{k^2}. \tag{13}$$

Note that the sign for the square root branch of the dispersion relation (11) is implicit in the above definitions of group velocity.

[1] Terms generated by harmonic coupling contribute at $O(\epsilon^4)$ in (7). Subsequent calculations (26) show that $B = O(\epsilon^3)$, hence the mean buoyancy corrections also contribute at $O(\epsilon^4)$ in (7).

To include higher-order corrections into modulation theory, we anticipate an adjustment of the frequency ω through the introduction of an $O(\epsilon^2)$ correction to the leading-order operator $\mathcal{M}$. A regularizing correction R now appears in the modified matrix

$$\mathcal{M}_R = \begin{pmatrix} k & -\omega & 0 \\ -\omega & 0 & -k \\ 0 & -k & -k\,(k^2+m^2+\epsilon^2 R) \end{pmatrix}. \tag{14}$$

The regularization strategy is based upon the method of Poincaré-Lindstedt, where the frequency imbedded in the leading-order operator is expanded to have deferred corrections [24].

The slow-scale equation (7) becomes

$$[\mathcal{M}_R + \epsilon\mathcal{L}_1 + \epsilon^2\mathcal{L}_{2R}]\mathbf{A_1} = O(\epsilon^4), \tag{15}$$

where the regularization term is added back into in the $O(\epsilon^2)$ operator

$$\mathcal{L}_{2R} = \begin{pmatrix} 0 & 0 & -kU_{\zeta\zeta} \\ 0 & 0 & 0 \\ 0 & 0 & k\,(\partial_\zeta^2 + R) \end{pmatrix}. \tag{16}$$

Consider a perturbation expansion for the vector amplitude $\mathbf{A_1}$:

$$\mathbf{A_1} = \mathbf{A_1^{(0)}} + \epsilon\mathbf{A_1^{(1)}} + \epsilon^2\mathbf{A_1^{(2)}} + O(\epsilon^3).$$

Plug this perturbation expansion into (15).

The zero-determinant condition on $\mathcal{M}_R$ leads to the following corrected dispersion relation:

$$\omega^2 = \frac{k^2}{k^2+m^2+\epsilon^2 R}. \tag{17}$$

Using (17), we can write $\mathcal{M}_R$ from (14) as

$$\mathcal{M}_R = \begin{pmatrix} k & -\omega & 0 \\ -\omega & 0 & -k \\ 0 & -k & -k^3/\omega^2 \end{pmatrix}. \tag{18}$$

At leading-order, the vector amplitude has the direction of the nullvector $\mathbf{a}_N$ of $\mathcal{M}_R$, but it is arbitrary up to a slowly-varying, real valued amplitude $A(\zeta,\tau)$:

$$\mathbf{A_1^{(0)}} = A\mathbf{a}_N = A\begin{pmatrix} 1 \\ k/\omega \\ -\omega/k \end{pmatrix}. \tag{19}$$

Since the procedure we follow is very similar to that developed in [8], we defer its details to the Appendix.

First, from the solvability condition at next order, a transport equation for the amplitude $A(\zeta,\tau)$ is derived — see (36). Introduce a scaled wave action $\mathcal{F}$, defined as

$$\mathcal{F} = A^2/\omega = \frac{(k^2+m^2)\gamma_1^2 + \alpha_1^2}{2\omega} \sim \frac{2}{\epsilon^2}\,\overline{\frac{\tilde{u}^2+\tilde{w}^2+\tilde{b}^2}{2\omega}} \tag{20}$$

which, by (3) and (19), is consistent with an equipartition of (horizontally averaged) wave energy density [5, 3]. Upon use of an integrating factor and the identification of the group velocity (12), the transport equation leads to the wave action equation:

$$\mathcal{F}_\tau + (c\mathcal{F})_\zeta = 0. \tag{21}$$

Calculations at second order yield expression (41) (see the Appendix) for the regularizing term R. Replacing the time derivatives by spatial derivatives (using the phase and transport equations) we can rewrite R as

$$\begin{aligned} R(m,\omega,A,U) = \frac{3}{4k^2}\frac{1}{\omega A}(2m\partial_\zeta + m_\zeta)\left\{\omega^2(2m\partial_\zeta + m_\zeta)(\omega A)\right\} + \frac{m}{2k^2}(m\omega\omega_\zeta)_\zeta + \frac{1}{4k^2}\left(m\omega_\zeta\right)^2 \\ - \frac{(\omega A)_{\zeta\zeta}}{\omega A} - \frac{k}{\omega}U_{\zeta\zeta}. \end{aligned} \tag{22}$$

To illustrate the mean flow contribution to the regularization term, we write (22) as

$$R(m,\omega,A,U) = R_{Lin}(m,\omega,A) - \frac{k}{\omega}U_{\zeta\zeta},$$

where R_{Lin} is the regularization term derived in [8] for the dispersive corrections of the linear Boussinesq system. The most significant nonlinear contribution to the regularization appears implicitly in the dependences of R_{Lin} on the modulation variables whose $O(\epsilon^2)$ dynamics now include coupling to the mean flow.

Mean equations. The mean flow equation for the horizontal velocity is typically derived by averaging over one period, in the x-variable, the horizontal momentum equation. Alternatively, it arises naturally in modulation theory as the x-independent mean terms obtained when the perturbation expansion (3) is substituted into (2)

$$\epsilon\frac{ik}{4}(\beta_1\gamma_1^* - \beta_1^*\gamma_1)_\zeta + O(\epsilon^3) = -U_{\tau\zeta}, \tag{23a}$$

$$\epsilon^2\frac{ik}{4}(\alpha_1\gamma_1^* - \alpha_1^*\gamma_1)_\zeta + O(\epsilon^4) = -B_\tau. \tag{23b}$$

It is noteworthy that waves can also induce changes to the mean stratification, although the effect is asymptotically weaker relative to the mean flow coupling. In addition, the quadratic terms generate only real contributions due to conjugate symmetry.

It is necessary to expand α_1, β_1 and γ_1 to first correction using (19) and (38) to obtain

$$\beta_1\gamma_1^* - \beta_1^*\gamma_1 = \epsilon\, 2i\left(\frac{A^2}{\omega}\right)_\tau + O(\epsilon^2) \tag{24a}$$

$$\alpha_1\gamma_1^* - \alpha_1^*\gamma_1 = \epsilon\,\frac{i}{k}\left(A^2\right)_\tau + O(\epsilon^2). \tag{24b}$$

In the above, the O(1) terms have cancelled by their being real-valued, and hence the nonlinear effects enter one order higher than might be expected.

To obtain the mean flow equation as a conservation law, (24a) is expressed in terms of the wave action $\mathcal{F}$ and the wave action equation (21). Upon ζ-integration, the equation governing the evolution of the horizontal mean flow to $O(\epsilon^2)$ is

$$U_\tau + \epsilon^2 \frac{k}{2} \left(c\mathcal{F}\right)_\zeta = 0. \tag{25}$$

The above mean flow equation, with the wave action equation (21), indicates that $U - \epsilon^2 \frac{k}{2}\mathcal{F}$ is a time-independent conserved quantity. For a Boussinesq fluid, this is the statement of conservation of (horizontal, Eulerian) momentum, which includes the wave-induced, $O(\epsilon^2)$ pseudomomentum [25, 10, 26]. The mean flow equation is commonly satisfied by the slaving relation $U = \epsilon^2 \frac{k}{2}\mathcal{F}$, which defines the wave-induced flow in the absence of background flow [27, 28]. The conservation law form is more general, as it allows for independent initialization of background horizontal shear, U, and wave amplitude, $\mathcal{F}$, as is done in the computations of Section 5.

However, we choose to satisfy the mean buoyancy equation (23b) by the slaving relation

$$B = -\epsilon^3 \frac{1}{4} \left(A^2\right)_\zeta, \tag{26}$$

which is the wave-induced, and adiabatic, heating/cooling as described in McIntyre [25]. This effect occurs at $O(\epsilon^3)$ so that this buoyancy correction decouples from the wave-meanflow interaction, and substantiates their neglect in the expansion (7)[1].

4 A system that may change type

The corrected modulation theory developed in the previous section consists of the equation of conservation of phase (6), the conservation of wave action (21) and the equation for the mean flow (25). We gather them in the following system

$$m_\tau + (\omega + kU)_\zeta = 0, \tag{27a}$$

$$\mathcal{F}_\tau + \left(c\mathcal{F}\right)_\zeta = 0, \tag{27b}$$

$$U_\tau + \epsilon^2 \frac{k}{2} \left(c\mathcal{F}\right)_\zeta = 0. \tag{27c}$$

The regularization is an $O(\epsilon^2)$ correction to the dispersion relation (17). The above system (27) is completed by the following expressions for reduced frequency and group velocity

$$\omega^2 = \frac{k^2}{k^2 + m^2 + \epsilon^2 R(m, \Omega(m), A, U)} \tag{28}$$

$$c = C(m) = -\frac{m\Omega^3}{k^2} \tag{29}$$

where R is given by (22) with

$$A^2 = \Omega(m)\,\mathcal{F}. \tag{30}$$

Note that the same square root branch is taken for both ω and Ω. This asymptotic truncation of the modulation equations produces an additional order of accuracy in the physical fields (b, η, ψ) over the standard (unregularized) system (4).

Without the regularization term ($R = 0$), the modulation equations (27) form a first-order PDE system whose characteristics are determined by the eigenvalues of the matrix

$$\begin{pmatrix} C & 0 & k \\ C'\mathcal{F} & C & 0 \\ \epsilon^2 \frac{k}{2} C'\mathcal{F} & \epsilon^2 \frac{k}{2} C & 0 \end{pmatrix}$$

where $C' = \frac{\partial C}{\partial m}$. The roots of the characteristic polynomial

$$\lambda \left[(\lambda - C)^2 - \epsilon^2 \frac{k^2}{2} C'\mathcal{F} \right] \tag{31}$$

show that the system is hyperbolic (real eigenvalues) when $C'\mathcal{F} > 0$, but weakly elliptic if $C'\mathcal{F} < 0$. This change of type is clearly a consequence of the mean flow coupling. Despite that the coupling is only an $O(\epsilon^2)$ perturbative effect, this dramatic change of type occurs via the breaking of the degeneracy of the double characteristic roots. The appearance of $\lambda = 0$ as an eigenvalue reflects the conservation of the total horizontal momentum. Using (11) and (29) we calculate

$$\begin{aligned} C'(m) &= = -\frac{\Omega^3}{k^2} \left(1 - \frac{3m^2}{k^2 + m^2} \right) \\ &= \frac{\Omega^3 (2m^2 - k^2)}{k^2 (k^2 + m^2)}. \end{aligned} \tag{32}$$

Hence, from (30), we get

$$C'\mathcal{F} = \frac{A^2 \Omega^2 (2m^2 - k^2)}{k^2 (k^2 + m^2)},$$

so that the system is seen to be weakly elliptic for long vertical waves $m^2 < k^2/2$.

Linear analysis of the truncated modulation system. We investigate the effect of the correction terms by performing a linear perturbation analysis for the truncated modulation system (27). Consider a (normalized) perturbation around a constant state $(m_0, \mathcal{F}_0, U_0)$:

$$m = m_0(1 + \mu), \quad \mathcal{F} = \mathcal{F}_0(1 + f), \qquad U = U_0(1 + u).$$

We also use

$$\omega = \omega_0(1 + \sigma), \quad \Omega = \omega_0(1 + \tilde{\sigma}) \quad \text{and} \quad A = A_0(1 + a),$$

where

$$\omega_0^2 = \frac{k^2}{k^2 + m_0^2} \quad \text{and} \quad A_0^2 = \omega_0 \mathcal{F}_0.$$

From (17), (11) and (30), we obtain by linearization:

$$\mu + \sigma \left(1 + \frac{k^2}{m_0^2} \right) + \frac{1}{2m_0^2} \epsilon^2 R = 0,$$

$$\mu + \tilde{\sigma} \left(1 + \frac{k^2}{m_0^2} \right) = 0,$$

and

$$2a = f + \tilde{\sigma}.$$

Also, from (22),

$$R(m,\Omega,A,U) \sim \frac{3}{2k^2} m_0^2\omega_0^2(\mu_{\zeta\zeta} + 2\tilde{\sigma}_{\zeta\zeta} + 2a_{\zeta\zeta}) + \frac{1}{2k^2} m_0^2\omega_0^2\tilde{\sigma}_{\zeta\zeta} - (\tilde{\sigma}_{\zeta\zeta} + a_{\zeta\zeta}) - \frac{k}{\omega_0} U_0 u_{\zeta\zeta}.$$

Based on the formulas (29) and (32), we make the notations

$$C_0 = -\frac{m_0\omega_0^3}{k^2} \qquad \text{and} \qquad C_0' = -\frac{\omega_0^3}{k^2} - \frac{3m_0\omega_0^2 C_0}{k^2}.$$

Finally, the linearization of the system (27) reads

$$\begin{aligned}
\mu_\tau &= -C_0\left(\mu + \epsilon^2\alpha\mu_{\zeta\zeta} + \epsilon^2\beta f_{\zeta\zeta} + \epsilon^2\gamma u_{\zeta\zeta}\right)_\zeta - \frac{kU_0}{m_0} u_\zeta \\
f_\tau &= -C_0\left(\frac{m_0 C_0'}{C_0}\mu + f\right)_\zeta \\
u_\tau &= -\epsilon^2 \frac{k}{2}\frac{C_0\mathcal{F}_0}{U_0}\left(\frac{m_0 C_0'}{C_0}\mu + f\right)_\zeta,
\end{aligned}$$

where coefficients α, β and γ of the regularizing terms, are given by

$$\begin{aligned}
\alpha &= -\frac{5}{2}\frac{\omega_0^4}{k^4} m_0^2 + \frac{3}{2}\frac{\omega_0^2}{k^2} \\
\beta &= -\frac{1}{4}\frac{C_0'}{m_0 C_0} \\
\gamma &= -\frac{1}{2}\frac{kU_0}{m_0^2\omega_0}.
\end{aligned}$$

Looking for solutions proportional to $e^{il(\zeta-\lambda\tau)}$ leads to the condition that λ is an eigenvalue of the matrix

$$C_0\begin{pmatrix} 1-\epsilon^2\alpha l^2 & -\epsilon^2\beta l^2 & \frac{kU_0}{m_0C_0} - \epsilon^2\gamma l^2 \\ \frac{m_0C_0'}{C_0} & 1 & 0 \\ \epsilon^2\frac{k}{2}\frac{\mathcal{F}_0}{U_0}\frac{m_0C_0'}{C_0} & \epsilon^2\frac{k}{2}\frac{\mathcal{F}_0}{U_0} & 0 \end{pmatrix}.$$

The eigenvalues are the roots of the characteristic polynomial

$$\lambda\left[(\lambda - C_0)^2 + \epsilon^2\alpha l^2 C_0(\lambda - C_0) - \epsilon^2\frac{k^2}{2}\mathcal{F}_0 C_0' + \left(\epsilon^4\frac{k}{2}\frac{\mathcal{F}_0}{U_0} m_0 C_0 C_0'\gamma + \epsilon^2 m_0 C_0 C_0'\beta\right) l^2\right].$$

With no regularizing terms ($\alpha, \beta, \gamma = 0$), the characteristic polynomial has real eigenvalues when $\mathcal{F}_0 C_0' > 0$, recovering the previous result regarding the stability (see (31)).

Replacing β and γ by their expressions, we find that the non-zero eigenvalues solve

$$(\lambda - C_0)^2 + \epsilon^2\alpha l^2 C_0(\lambda - C_0) - \epsilon^2\frac{k^2}{2}\mathcal{F}_0 C_0' + \left(-\epsilon^4\frac{k^2}{4}\frac{\mathcal{F}_0}{m_0\omega_0} C_0 C_0' - \epsilon^2\frac{1}{4}{C_0'}^2\right) l^2 = 0.$$

The eigenvalues are imaginary (implying instability) when the discriminant of the 2nd-order equation is negative:

$$\epsilon^2\alpha^2 C_0^2 l^4 + 2k^2\mathcal{F}_0 C_0' + {C_0'}^2 l^2 + \epsilon^2 k^2 \frac{\mathcal{F}_0}{m_0\omega_0} C_0 C_0' l^2 < 0.$$

Using

$$\frac{k^2}{m_0\omega_0 C_0} = -\frac{\omega_0^2}{C_0^2},$$

we can describe the instability region by

$$\epsilon^2\alpha^2 C_0^2 l^4 + {C_0'}^2 l^2 < -2k^2\mathcal{F}_0 C_0' + \epsilon^2\omega_0^2 l^2 \mathcal{F}_0 C_0'.$$

Consider the case $\mathcal{F}_0 C_0' < 0$, when the unregularized system $(\alpha, \beta, \gamma = 0)$ is linearly unstable for *all* wavenumbers l. Note that by adding the regularizing terms, the instability region is reduced to

$$\epsilon^2 \left(2k^2 - \epsilon^2\omega_0^2 l^2\right) |\mathcal{F}_0 C_0'| > \epsilon^2\alpha^2 C_0^2 l^4 + {C_0'}^2 l^2,$$

which is a subset of

$$0 < l^2 < \frac{2k^2}{\epsilon^2\omega_0^2}.$$

Therefore the added correction terms introduce a short-wave cut-off of the unstable wavenumbers, reducing them to a subset of the range specified above. As discussed in Section 2, similar stabilizing effect of the correction terms can be observed in the modulation theory for the Klein-Gordon equation (Chapter 15.5, [1]) and the focusing NLS equation (Chapter 2d, [23]).

5 Numerical results

The goals of this numerical study are a) to investigate the effect of adding the correction terms to the modulation system and b) to compare the solution of the modulation system against the corresponding variables from the original Boussinesq equations.

Therefore we solve numerically both the nonlinear Boussinesq equations (1) and the modulation system (27). For the latter, ω and c are given by (28) and (29), while the correction term R, the amplitude A and the leading-order frequency Ω are computed from (22), (30), and (11), respectively.

The numerical method we use is standard: the spatial derivatives are computed pseudo-spectrally and the time-stepping is done using the 4th-order Runge-Kutta method. The linear part in the Boussinesq equations is solved exactly using the integrating factor method.

We consider the following initial condition for (27):

$$m(\zeta, 0) = 1 \tag{33a}$$

$$A(\zeta, 0) = 0.5(1 + 0.05\cos\zeta)^4 \tag{33b}$$

$$U(\zeta, 0) = 1 - 0.03\sin\zeta, \tag{33c}$$

corresponding to a modulated monochromatic wavetrain. The initial data for the wave action $\mathcal{F}(\zeta, 0)$ is computed from (11) and (30). Also, using (33), we initialize the Boussinesq system according to (3).

The slow space variable ζ takes values in $[-\pi, \pi]$, while the fast variable $z = \frac{\zeta}{\epsilon}$ in $\left[-\frac{\pi}{\epsilon}, \frac{\pi}{\epsilon}\right]$. In the numerical experiments presented here, $\epsilon = 1/12$. Note that in all figures below the modulation and the Boussinesq variables are plotted against the fast variable z.

We show two sets of numerical results, corresponding to two values of k:

1. run 1: $k = 1$

2. run 2: $k = 2$

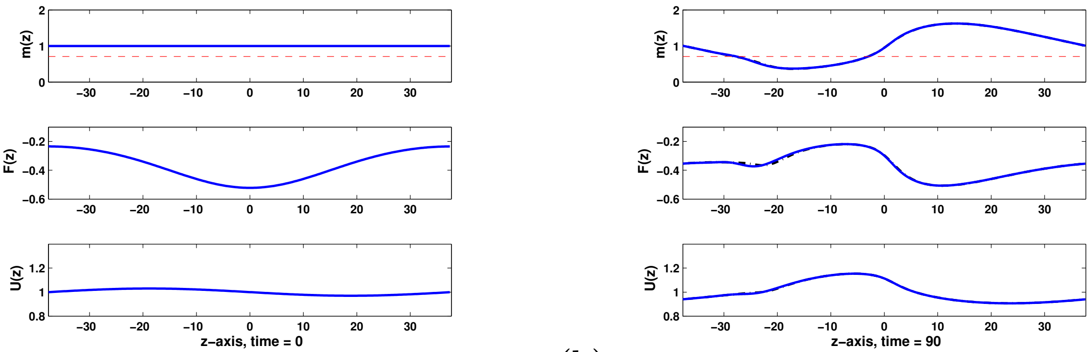
Run 1 corresponds to initial data entirely contained in the hyperbolic region of the unregularized modulation system, as $m(\zeta, 0) > k/\sqrt{2}$ for all ζ. Figure 1(a) shows the initial data (33) ($k = 1$), where part of the boundary of the hyperbolic-elliptic region ($m = k/\sqrt{2}$) is indicated by a horizontal dashed line in the top plot. Figure 2(a) shows the cross-section $x = 0$ of the initial buoyancy $b(z, 0)$ of the Boussinesq system. It consists of a monochromatic ($m = 1$) wavetrain whose envelope is given by $A(\zeta, 0)$ from (33b) [2] .

Figure 1(b) shows both the solution of the corrected modulation system (dash-dot line) and the modulation variables obtained from the Bossinesq equations (solid line) at $t = 90$. There is an excellent agreement between the two sets of solutions. In fact they can barely be distinguished from each other. The buoyancy $b(z, t)$ at $t = 90$ obtained from the Boussinesq equations is shown in Figure 2(b), along with its envelope (solid line on top) and the amplitude function $A(z, t)$ (dash-dot line) provided by the modulation theory. At $t \approx 54.09$ the modulation variables enter into the elliptic region, but no instability is observed in the solution of the corrected system. The elliptic region continues to grow and at $t = 90$ it represents more than a third of the entire domain, as can be seen in the top plot of Figure 1(b) — recall that the horizontal dashed line depicts the boundary of the hyperbolic-elliptic region. The illustration of the elliptic instability for the unregularized modulation system is depicted in Figure 3. Figure 3(a) shows the high wavenumber instability, associated with the system's change of type from hyperbolic to elliptic. Figure 3(b) illustrates the effect of the instability in the physical space. High frequency oscillations develop as soon as the system enters the elliptic region $m^2 < k^2/2$.

This numerical example illustrates how the regularization terms help in preventing the instability associated with the change of type of the leading-order system. Even more impressive is how the correction terms regularize the dynamics in run 2.

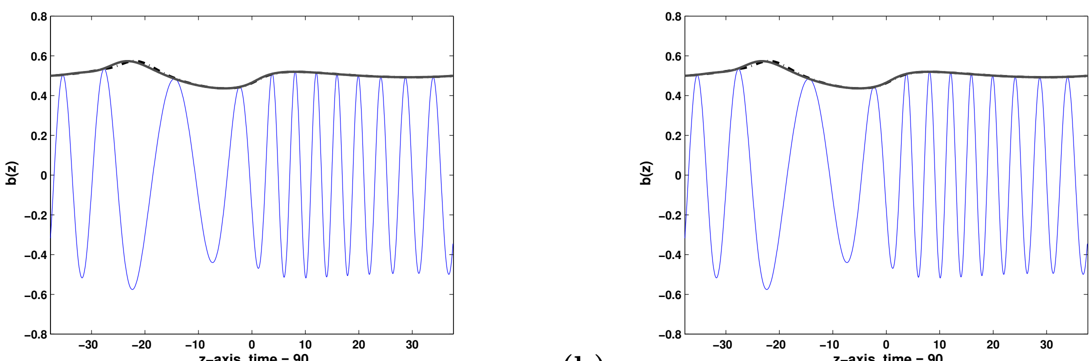
Run 2 corresponds to an initial condition that lies entirely in the elliptic domain of the unregularized system ($m(\zeta, 0)^2 < k^2/2$ for all ζ) — see the top plot of Figure 4(a), where the graph of m at $t = 0$ is under the dashed line representing the hyperbolic-elliptic boundary. The unregularized modulation system ($R = 0$) with such an initial condition is simply unstable for all wave-numbers and its solution would blow-up quickly. When the regularization terms are added however, the growth is stabilized and the modulation system produces results in excellent agreement with the Boussinesq equations. Figure 4(b) shows the solution of the modulation system at $t = 60$, compared against the solution of the Boussinesq system. The two sets of solutions are virtually indistinguishable. Note that $m(z, t)$ has crossed the boundary $m = k/\sqrt{2}$ during its time evolution. Once in the hyperbolic region, the solution steepens to form a shock, but the regularizing terms prevent the singularity formation through dispersion, as illustrated

[2] According to (3), the Boussinesq variables b and η have an extra factor of ϵ that expresses the small amplitude regime. We omit this factor to produce even more conclusive test cases for the regularized modulation theory developed in this article.



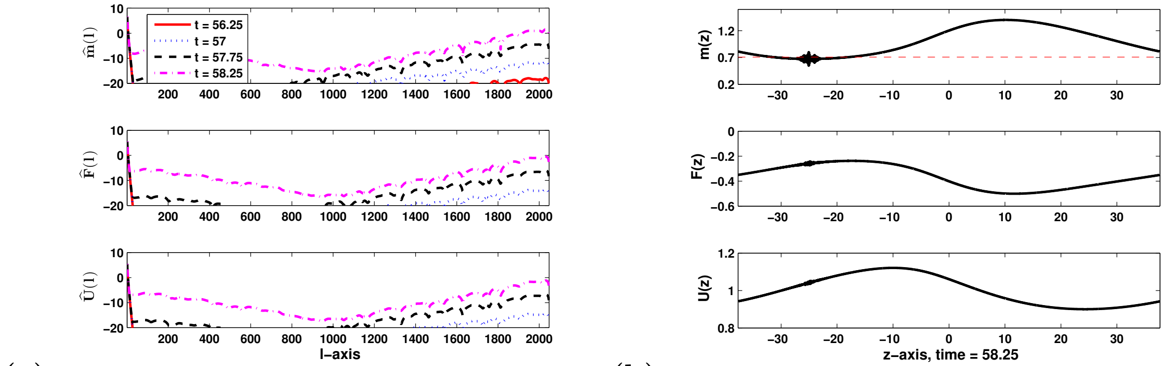
(a) (b)

Figure 1: Run 1: (a) Initial data (33) for $k = 1$, contained entirely in the hyperbolic region, as $m > k/\sqrt{2}$. (b) Modulation variables m, $\mathcal{F}$ and U obtained from the Boussinesq equations (solid line) and from the corrected modulation system (dash-dot line) at $t = 90$. The agreement is excellent, as the two sets of solutions can hardly be distinguished from each other. The dashed lines in the two top plots represent the hyperbolic-elliptic boundary $m = k/\sqrt{2}$.



(a) (b)

Figure 2: Run 1: (a) Cross section $x = 0$ of the initial buoyancy and its envelope. The initial data for the Boussinesq system is obtained from (3) — see footnote [2] — and (33), where $\epsilon = 1/12$. (b) Cross section $x = 0$ of the solution $b(z, t)$ of the Boussinesq equations and its envelope (solid line) at $t = 90$. The dashed line on top of the solution represents the amplitude $A(z, t)$ obtained by solving numerically the modulation equations (33).



(a) (b)

Figure 3: Run 1: Illustration of the elliptic instability for the unregularized modulation system (a) Fourier spectrum of m, $\mathcal{F}$ and U at $t = 56.25$ (solid line), shortly after the system changes from hyperbolic to mixed hyperbolic-elliptic type, $t = 57$ (dotted line), $t = 57.75$ (dashed-line) and $t = 58.25$ (dash-dotted line). (b) Modulation variables m, $\mathcal{F}$ and U obtained from the unregularized modulation system at $t = 58.25$. The instability is indicated by the high frequency oscillations present in the elliptic region. The dashed line in the top plot represents the hyperbolic-elliptic boundary $m = k/\sqrt{2}$.

in our work on the linear Boussinesq equations [8]. We do not present the shock regularization of the correction terms here.

6 Appendix

We present here details on the calculations leading to the wave action equation (21) and the expression (22) for the regularization term R.

The $O(\epsilon)$ terms in the slow-scale equation (15) give

$$\mathcal{M}_R\,\mathbf{A}_\mathbf{1}^{(\mathbf{1})} + \mathcal{L}_1(A\,\mathbf{a}_N) = 0, \tag{34}$$

which the correction $\mathbf{A}_\mathbf{1}^{(\mathbf{1})}$ must satisfy. As $\mathcal{M}_R$ is a (symmetric) singular matrix, existence of $\mathbf{A}_\mathbf{1}^{(\mathbf{1})}$ requires the solvability condition

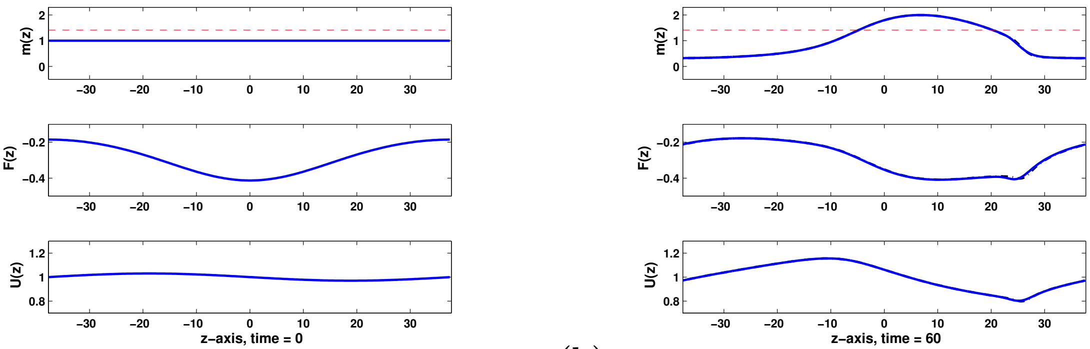
$$(\mathbf{a}_N)^{\mathrm{T}}\mathcal{L}_1(A\,\mathbf{a}_N) = 0 \tag{35}$$

and results in the scalar transport equation for the amplitude $A(\zeta,\tau)$

$$\partial_\tau\left(\frac{k}{\omega}A\right) + \frac{k}{\omega}\,\partial_\tau A - \omega\left\{2m\partial_\zeta + m_\zeta\right\}\left(\frac{\omega}{k}A\right) = 0. \tag{36}$$

By inspection, A is seen to be an integrating factor for this expression (since k is a constant), allowing a conservation law form

$$\left(\frac{A^2}{\omega}\right)_\tau - \left(\frac{m\omega^3}{k^2}\,\frac{A^2}{\omega}\right)_\zeta = 0. \tag{37}$$



(a) **(b)**

Figure 4: Run 2: (a) Initial data (33) for $k = 2$, contained entirely in the elliptic region $m^2 < k^2/2$. (b) Modulation variables m, $\mathcal{F}$ and U obtained from the Boussinesq equations (solid line) and from the corrected modulation system (dash-dot line) at $t = 60$. The two sets of solutions are virtually indistinguishable from each other. The dashed lines in the two top plots represent the hyperbolic-elliptic boundary $m = k/\sqrt{2}$. Note that a solution of the leading-order modulation system ($R = 0$) with such an initial condition would blow up quickly due to the elliptic instability.

Equation (37) can be written in terms of the wave action $\mathcal{F}$ to provide (21).

Continuing to next order in the perturbation requires an explicit representation of $\mathbf{A_1^{(1)}}$. We showed in [8] that a solution of (34) that enables major simplifications in the calculations of the regularizing term R is provided by

$$\mathbf{A_1^{(1)}} \;=\; = \; -i \begin{pmatrix} -\dfrac{1}{2}\left(\dfrac{A}{\omega}\right)_\tau \\ \dfrac{k}{2\omega}\left(\dfrac{A}{\omega}\right)_\tau \\ \dfrac{1}{k}A_\tau + \dfrac{\omega}{2k}\left(\dfrac{A}{\omega}\right)_\tau \end{pmatrix} \tag{38}$$

The $O(\epsilon^2)$ terms in the slow-scale equation (15) yield

$$\mathcal{M}_R\,\mathbf{A_1^{(2)}} + \mathcal{L}_1(\mathbf{A_1^{(1)}}) + \mathcal{L}_{2R}(A\,\mathbf{a}_N) = 0, \tag{39}$$

from which the solvability condition for the existence of $\mathbf{A_1^{(2)}}$,

$$(\mathbf{a}_N)^{\mathrm{T}}\left\{\mathcal{L}_1(\mathbf{A_1^{(1)}}) + \mathcal{L}_{2R}(A\,\mathbf{a}_N)\right\} = 0, \tag{40}$$

determines R.

The solvability condition (40) becomes

$$-\frac{k}{2}\left(\frac{1}{\omega}\right)_\tau\left(\frac{A}{\omega}\right)_\tau - \frac{\omega}{k}(2m\partial_\zeta + m_\zeta)\left(A_\tau + \frac{\omega}{2}\left(\frac{A}{\omega}\right)_\tau\right) + U_{\zeta\zeta}A\omega + \frac{\omega}{k}\partial^2_{\zeta\zeta}(\omega A) + \frac{1}{k}\omega^2 AR = 0.$$

Solving for R we obtain

$$R = \frac{1}{\omega A}(2m\partial_\zeta + m_\zeta)\left(A_\tau + \frac{\omega}{2}\left(\frac{A}{\omega}\right)_\tau\right) + \frac{k^2}{2}\frac{1}{\omega^2 A}\left(\frac{1}{\omega}\right)_\tau\left(\frac{A}{\omega}\right)_\tau - \frac{1}{\omega A}(\omega A)_{\zeta\zeta} - \frac{k}{\omega}U_{\zeta\zeta}. \quad (41)$$

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References

[1] G. B. Whitham, *Linear and Nonlinear Waves.* Wiley, 1974.

[2] P. G. Drazin, “Non-linear internal gravity waves in a slightly stratified atmosphere,” *J. Fluid Mech.* **36** (1969) 433–446.

[3] R. Grimshaw, “Nonlinear internal gravity waves in a slowly varying medium,” *J. Fluid Mech.* **54** (1972) 193–207.

[4] B. S. H. Rarity, “A theory of the propagation of internal gravity waves of finite amplitude,” *J. Fluid Mech.* **39** (1969) 497–509.

[5] F. P. Bretherton, “On the mean motion induced by internal gravity waves,” *J. Fluid Mech.* **36** (1969) 785–803.

[6] F. P. Bretherton, “The generalised theory of wave propagation,” in *Mathematical Problems in the Geophysical Sciences, 1. Geophysical Fluid Dynamics, W. H. Reid ed.*, vol. 13 of *Lectures in Applied Mathematics*, pp. 61–102. American Mathematical Society, Providence, RI, 1971.

[7] R. Grimshaw, “Nonlinear internal gravity waves and their interaction with the mean wind,” *J. Atmos. Sci.* **32** (1975) 1779–1793.

[8] R. F. Fetecau and D. J. Muraki, “Dispersive corrections to a modulation theory for stratified gravity waves,” *Wave Motion* (2010) doi:10.1016/j.wavemoti.2010.02.003.

[9] P. D. Lax and C. D. Levermore, “The small dispersion limit of the Korteweg-deVries equation i,” *Comm. Pure Appl. Math.* **36** (1983) 253–290.

[10] B. R. Sutherland, “Weakly nonlinear internal gravity wavepackets,” *J. Fluid Mech.* **569** (2006) 249–258.

[11] A. Tabaei and T. R. Akylas, “Resonant long-short wave interactions in an unbounded rotating stratified fluid,” *Stud. Appl. Math.* **119** (2007) 271–296.

[12] V. I. Shrira, "On the propagation of a three-dimensional packet of weakly non-linear internal gravity waves," *Int. J. Non-Linear Mechanics* **16** (1981) no. 2, 129–138.

[13] J. C.Luke, "A perturbation method for nonlinear dispersive waves," *Proc. Roy. Soc. A* **292** (1966) 403–412.

[14] F. P. Bretherton, "Propagation in slowly varying waveguides," *Proc. Roy. Soc. A* **302** (1968) 555–576.

[15] G. B. Whitham, "Two-timing, variational principles and waves," *J. Fluid Mech.* **44** (1970) 373–395.

[16] J. R. Holton and R. S. Lindzen, "An updated theory for the quasi-biennial cycle of the tropical stratosphere," *J. Atmos. Sci.* **29** (1972) 1076–1080.

[17] R. A. Plumb, "The interaction of two internal wave with the mean flow: Implications for the theory of the quasi-biennial oscillation," *J. Atmos. Sci.* **35** (1977) 1847–1858.

[18] T. J. Dunkerton, "The role of gravity waves in the quasi-biennial oscillation," *J. Geophys. Res.* **102** (1997) 26,053–26,076.

[19] M. Baldwin, L. Gray, T. Dunkerton, K. Hamilton, P. Haynes, W. Randel, J. Holton, M. Alexander, I. Hirota, T. Horinouchi, D. Jones, J. Kinnersly, C. Marquardt, K. Sato, and M. Takahashi, "The quasi-biennial oscillation," *Reviews of Geophysics* **39** (2001) 179–229.

[20] R. A. Plumb and D. McEwan, "The instability of a forced standing wave in a viscous stratified fluid: A laboratory analogue of the quasi-biennial oscillation," *J. Atmos. Sci.* **35** (1978) 1827–1839.

[21] N. P. Wedi and P. K. Smolarkiewicz, "Direct numerical simulation of the plumb-mcewan laboratory analog of the qbo," *J. Atmos. Sci.* **63** (2006) 3226–3252.

[22] V. H. Chu and C. C. Mei, "On slowly-varying Stokes waves," *J. Fluid Mech.* **41** (1970) 873–887.

[23] A. C. Newell, *Solitons in Mathematics and Physics.* SIAM, 1985.

[24] J. Kevorkian and J. D. Cole, *Multiple Scale and Singular Perturbation Methods.* Springer-Verlag, New York, 1996.

[25] M. E. McIntyre, "An Introduction to the Generalized Lagrangian-Mean Description of Wave, Mean-Flow Interaction," *Pageoph* **118** (1980) 151–175.

[26] O. Bühler, "Wave-vortex interactions in fluids and superfluids," *Annu. Rev. Fluid Mech.* **42** (2010) 205–228.

[27] R. Grimshaw, "Internal gravity waves in a slowly varying, dissipative medium," *Geophys. Fluid. Dyn.* **6** (1974) 131–148.

[28] B. R. Sutherland, "Internal wave instability: Wave-wave versus wave-induced mean flow interactions," *Phys. Fluids* **18** (2006) no. 7, 074107, 8.