ON MINIMIZERS OF INTERACTION FUNCTIONALS WITH COMPETING ATTRACTIVE AND REPULSIVE POTENTIALS

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ABSTRACT. We consider a family of interaction functionals consisting of power-law potentials with attractive and repulsive parts and use the concentration compactness principle to establish the existence of global minimizers. We consider various minimization classes, depending on the signs of the repulsive and attractive power exponents of the potential. In the special case of quadratic attraction and Newtonian repulsion we characterize in detail the ground state.

1. Introduction

We consider the minimization of energies of the form

$$E[\rho] := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\rho(x)\rho(y)\,dxdy, \tag{1.1}$$

where

$$K(x) := \frac{1}{q}|x|^q - \frac{1}{p}|x|^p, \qquad \text{for } -N < p < q. \tag{1.2}$$

These functionals are directly connected to a class of self-assembly/aggregation models which recently have received much attention (see for example, [9, 11, 12, 19, 33, 36, 38, 39, 40, 47, 54, 58, 59]). The aggregation models consist of the following active transport equation in $\mathbb{R}^N$ for the population density ρ:

$$\rho_t + \nabla\cdot(\rho\mathbf{v}) = 0, \quad \mathbf{v} = -\nabla K * \rho, \tag{1.3}$$

where K represents the interaction potential and $*$ denotes spatial convolution. This partial differential equation is the gradient flow of the energy (1.1) with respect to the 2-Wasserstein metric [1, 26]. Indeed, the evolution equation (1.3) can be written in the form

$$\partial_t\rho = \nabla\cdot\left(\rho\nabla\frac{\delta E[\rho]}{\delta\rho}\right),$$

which is the standard form for the 2-Wasserstein gradient flow [1] of the energy (1.1).

Model (1.3) appears in the study of many phenomena, including biological swarms [54, 58], granular media [9, 59], self-assembly of nanoparticles [38, 39] and molecular dynamics simulations of matter [37]. The study of solutions to (1.3) (well-posedness, finite or infinite time blow-up, long-time behavior) has been a very active area of research during the past decade [11, 12, 19, 33, 47]. It is important to note that the analysis and behavior of solutions to (1.3) depend essentially on the properties of the potential K. In the context of biological swarms, K

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incorporates social interactions (attraction and repulsion) between group individuals. Potentials which are attractive in nature typically lead to blow-up [11, 40], while attractive-repulsive potentials may generate finite-size, confined aggregations [36, 47].

By inspecting the equation for $\mathbf{v}$ in (1.3) one notes that the nature of a symmetric potential $K(x) = K(|x|)$ is dictated by the sign of its derivative ($K' > 0$ corresponds to attraction and $K' < 0$ to repulsion). Hence, for K given by (1.2), the exponent q refers to attraction and p to repulsion (p and q can be of any sign). The condition $q > p$ is needed to ensure that the potential is repulsive at short ranges and attractive in the far field — see Figures 1(a)–1(c) for a generic illustration of K with $p < 0$ and Figure 1(d) for an example of K with $p > 0$. Note that in the regime $p < 0$ when $q \geqslant 1$, the potential K is positive, convex and $K \to \infty$ as $|x| \to \infty$ whereas when $0 < q < 1$, K is still positive and grows indefinitely with $|x|$; however, it is not convex. Finally for $-N < q < 0$, K becomes negative, approaches 0 as $|x| \to \infty$ and is not convex.

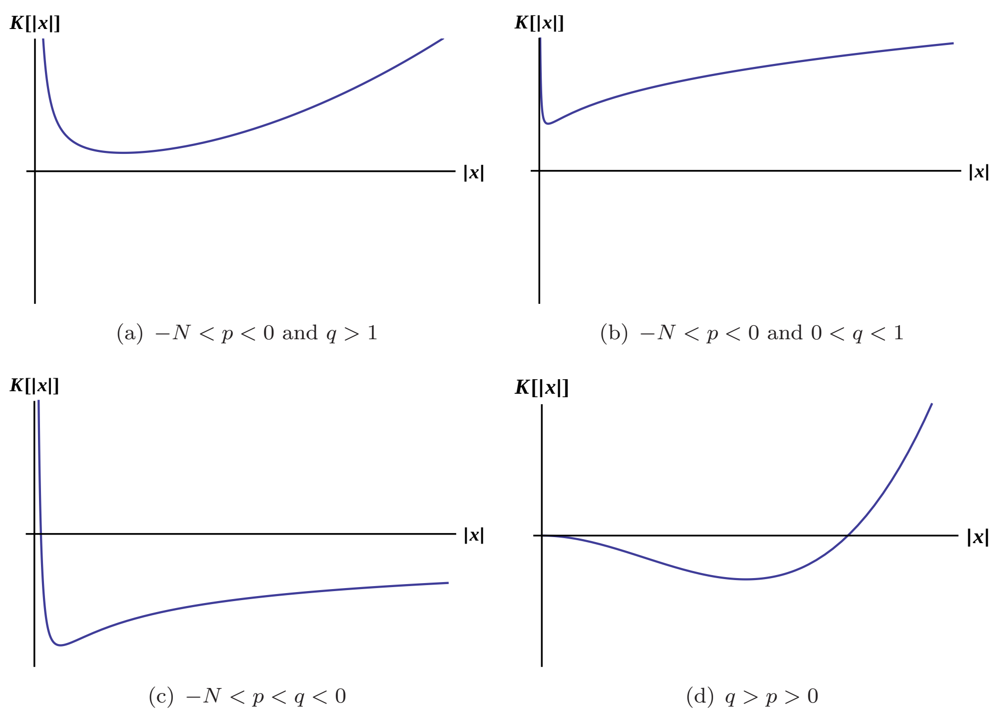
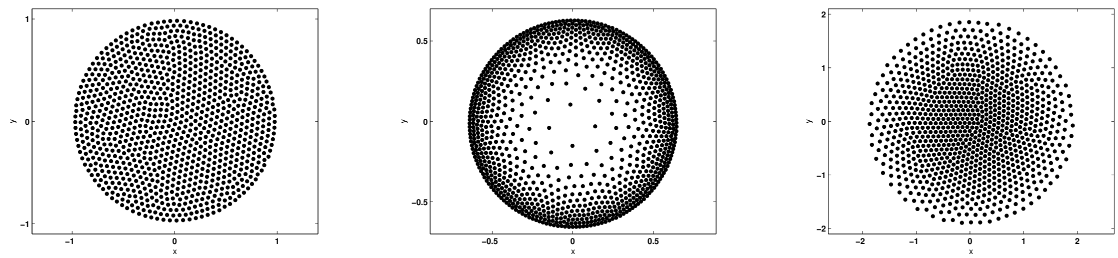


FIGURE 1. Generic examples of K for various values of p and q.

Potentials in power-law form have been frequently considered in the recent literature on the aggregation model (1.3) [5, 35, 36, 44, 60]. As shown in these works, the delicate balance between attraction and repulsion often leads to complex equilibrium configurations, supported on sets of various dimensions. Indeed, a simple particle model simulation in two dimensions shows accumulation of the density in different states depending on the powers of the interaction potential K (see Figures 2 and 3). The dimensionality of local minimizers of (1.1) with K given by (1.2) was recently investigated in [4]. The repulsion exponent p in [4] is restricted to be above the Newtonian singularity, i.e., $p > 2 - N$.

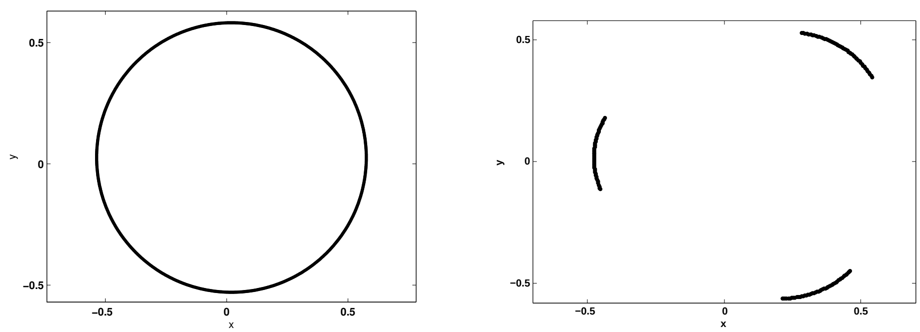


(a) Newtonian repulsion, quadratic attraction

(b) Newtonian repulsion, large positive attraction

(c) Below Newtonian repulsion, small negative attraction

FIGURE 2. 2D Particle model simulations in the regime $p < 0$. In 2D, the Newtonian repulsion is given by $-\log|x|$ and below Newtonian repulsion corresponds to $-2 < p < 0$ (see also Remark 3.5).



(a) $q = 10$, $p = 1$

(b) $q = 7$, $p = 1.5$

FIGURE 3. 2D Particle model simulations in the regime $q > p > 0$.

A significant number of recent works exploited the gradient flow structure in the particle (individual-based) model associated to (1.3). Specifically, the PDE model (1.3) can be regarded as the continuum limit of the following particle model describing the pairwise interaction of M particles in $\mathbb{R}^N$ [18]:

$$\frac{dX_i}{dt} = -\frac{1}{M} \sum_{\substack{j=1 \\ j \neq i}}^{M} \nabla K(X_i - X_j), \qquad i = 1 \dots M, \tag{1.4}$$

where $X_i(t)$ represents the spatial location of the i-th individual at time t. The particle model (1.4) is the gradient flow of the interaction energy which is the discrete version of (1.1) [44]. It has been shown that simple choices of interaction potentials in (1.4) can lead to very diverse and complex equilibrium solutions (e.g., disks, rings and annular regions in 2D, balls, spheres and soccer balls in 3D) [36, 44, 60].

By staying entirely at the continuum level (that is, working with (1.3) without resorting to the particle system (1.4)) it is more difficult to identify equilibria. There are only a few works in this direction. In [35, 36] the authors study equilibrium solutions to (1.3) which are supported in a ball, while in [5] the focus is equilibria that are uniformly distributed on spherical shells. Such equilibria, along with those revealed by simulations of the discrete model, constitute the main motivation for this work. In this article, we directly study the problem from the variational point of view, i.e., minimizers of the nonlocal energy (1.1). In particular, we show the existence of a global minimizer of (1.1) over the classes of uniformly bounded functions (for $p < 0$ and $q > 0$), radially symmetric and uniformly bounded functions (for $p < 0$ and $q < 0$), and probability measures (for $p > 0$). We address and motivate the assumptions of radial symmetry and uniform boundedness in the summary below (see Section 2 and Remark 3.6). Also, most of the previous works referenced above consider power-law potentials where the repulsion-power is assumed to be above Newtonian, that is, $p \geqslant 2 - N$. However, in our results we only need integrability at the origin, and consequently, p can take values in the larger range $p > -N$.

Here we also would like to note that during the review process of this manuscript we noticed that the radial symmetry and uniform boundedness assumption have been relaxed in the recent preprints [28, 25] in certain regimes of p and q-values of power-law potentials whereas the results in [24, 56] extend our existence results to more general potentials.

We conclude the introduction with two remarks concerning related problems.

Remark 1.1 (Repulsion via nonlinear diffusion)**.** A model related to (1.3) considers an interaction kernel which is purely attractive and incorporates repulsive interactions through nonlinear diffusion. This model reads

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = \Delta \rho^m, \qquad \mathbf{v} = -\nabla K * \rho, \tag{1.5}$$

where K is purely *attractive* and $m > 1$ is a real exponent (cf. [7, 8, 14, 17, 20, 21, 22, 23]). Here the associated energy functional is given by

$$E_{\mathrm{nld}}[\rho] := \frac{1}{2} \int \int K(x-y)\rho(x)\rho(y)dxdy + \frac{1}{m-1} \int \rho^m(x)\, dx.$$

Using Lions' concentration compactness lemma, a proof of global existence for this functional was given in [6]. A detailed study of steady state solutions of (1.5) in one dimension is given in [22, 23]. Note that these results do not need a uniform L^∞-bound on the admissible densities as the energy E_{nld} controls some L^s-norm of the density function ρ. Also, since the interaction term in the energy E_{nld} is purely attractive, using symmetric decreasing rearrangement type arguments one sees that the minimizers have to be radially symmetric.

Similar energies also appear in astrophysics and quantum mechanics and have been extensively studied [2, 3, 52]. In fact, in the seminal paper of Lions [52] wherein he introduces the concentration compactness lemma, a direct application is the existence of minimizers to a class of these L^1 minimization problems. Many of the arguments in our present article follow his application.

Remark 1.2 (Thomas-Fermi-Dirac-von Weizsäcker Functionals and the Nonlocal Isoperimetric Problem)**.** Many other functions with interacting attractive and repulsive components have been studied, for example the Thomas-Fermi-Dirac-von Weizsäcker functional in mathematical physics [45, 46, 49, 53]. Recently, a binary *nonlocal isoperimetric functional* appeared in connection with the modeling of self-assembly of diblock copolymers [29, 30, 31]. Existence and non-existence results have been presented in [41, 43, 42, 53]. In dimension $N = 3$, the functional has a Newtonian repulsive component as in (1.1) with $p = -1$. However, the attractive component does not come from an interaction term but rather by adding a higher-order regularization. Precisely, for $m > 0$, the nonlocal isoperimetric problem is to minimize

$$E_{\text{nlip}}[\rho] := \int_{\mathbb{R}^3} |\nabla \rho| \; + \; \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho(x)\rho(y)}{4\pi|x-y|}\, dx\, dy$$

over

$$\rho \in BV(\mathbb{R}^3, \{0,1\}) \;\text{ with }\; \int_{\mathbb{R}^3} \rho \,=\, m.$$

Since admissible densities ρ are characteristic functions, the first term in the energy is simply the perimeter of the support. Not only is there a competition between the two terms in E_{nlip} but they are in *direct competition* in the following sense: balls are *best* (least energy) for the first (attractive) term and *worst* (greatest energy) for the second (repulsive) term. The latter point has an interesting history. Poincaré [55] considered the problem of determining the equilibrium shapes of a fluid body of mass m. In simplified form, this amounts to minimizing the total potential energy of the region of fluid $E \subset \mathbb{R}^3$

$$\int_E \int_E -\frac{1}{C\,|x-y|}\, dx\, dy,$$

where $-(C|x-y|)^{-1}$ $(C > 0)$ is the potential resulting from the gravitational attraction between two points x and y in the fluid. Poincaré showed that, under some smoothness assumptions, a body has the lowest energy if and only if it is a ball. It was not until almost a century later that the essential details were sorted via the rearrangement ideas of Steiner for the isoperimetric inequality. These ideas are captured in the *Riesz Rearrangement Inequality* and its development (cf. [48, 51]).

In [29, 30], it was conjectured that there exists a critical mass m_0 below which, a unique minimizer of E_{nlip} exists and is the characteristic function of a ball, and above which, the minimizer fails to exist because of "mass" escaping to infinity. Note this is in stark contrast to minimizers of (1.1). The non-existence for sufficiently large m has recently been proved in [53]. Existence of a radially symmetric minimizer (i.e. a ball) for m sufficiently small has recently been proved in [41, 42]. Whether or not balls are the *only* minimizers remains open.

2. Statements of Our Results

In this section we state and summarize our main results, placing them in the context of other recent work.

2.1. Existence of Global Minimizers.

We consider the existence of minimizers in two separate cases: $p < 0$ and $p > 0$.

Negative power repulsion $p < 0$. Figures 2 (a)-(c) show examples of particle simulations for interaction potentials that consist of Newtonian repulsion[1] and an attraction component with various powers q. In the case of negative p, similar simulations show that particles do not accumulate on lower-dimensional sets. For example, in [36], the time-dependent density $\rho(\cdot, t)$ is shown to be uniformly bounded in L^∞ for all $t > 0$ provided the initial condition

[1] In $\mathbb{R}^N$, the Newtonian potential is given by the repulsive part of (1.2) with $p = 2 - N$. For 2-D particle simulations we use $-\log|x|$ as the Newtonian repulsion. See Remark 3.5 for details.

ρ_0 is in L^∞. Also in [4], the authors prove that for any N and $-N < p < 0$, minimizers do not accumulate on a set of dimension less than $2 - p$ and point out that they never observed minimizers with support of non-integer Hausdorff dimension. This means that when $N = 3$ the minimizers are indeed functions; however, for $N > 3$ the result is weaker and only gives a lower bound. Nonetheless, for $p < 0$, we expect that minimizers exist in the space of density *functions* (and not measures). As a matter of fact, we immediately see that a Dirac delta integrated against an interaction potential $1/|x|^a$ with $0 < a < N$ cannot be a minimizer of the energy (2.1). However, with only an L^1-bound on the density, accumulation along a set of Hausdorff dimension less than N is a possibility – in fact as we discuss below, such possibilities are indeed generic when $p > 0$. For $p < 0$, even though there might be some minor symmetry defects depending on the choice of number of particles, the simulations with random initial data suggest that the steady states are radially symmetric.

We thus consider the minimization of the energy E in the class of radially symmetric, uniformly bounded densities when the attraction-power q is negative. However, we relax the radial symmetry assumption for positive exponents $q > 0$. To this end, for $p < 0$ we take our admissible class of densities ρ as the space of non-negative, uniformly bounded L^1 functions with fixed mass m when $q > 0$, and as the space of non-negative, radially symmetric, uniformly bounded L^1 functions with fixed mass m when $q < 0$. That is, for $-N < p < 0$, $q > p$, $m > 0$ and $M > 0$, we consider the following variational problem:

$$\text{minimize} \quad E(\rho) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(\frac{1}{q}|x-y|^q - \frac{1}{p}|x-y|^p\right)\rho(x)\rho(y)\,dxdy \tag{2.1}$$

over

$$\mathcal{A} := \left\{\rho \in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N): \quad \rho \geqslant 0,\ \|\rho\|_{L^\infty} \leqslant M, \text{ and } \int_{\mathbb{R}^N}\rho(x)\,dx = m\right\} \tag{2.2}$$

when $q > 0$; and over

$$\mathcal{A}_r := \left\{\rho \in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N): \quad \rho = \rho(|x|) \geqslant 0,\ \|\rho\|_{L^\infty} \leqslant M, \text{ and } \int_{\mathbb{R}^N}\rho(x)\,dx = m\right\} \tag{2.3}$$

when $q < 0$.

For this minimization problem the first main result we obtain is the existence of minimizers.

Theorem 2.1. *There exists a minimum of* (2.1) *in $\mathcal{A}$ when $-N < p < 0 < q$, and in $\mathcal{A}_r$ when $-N < p < q < 0$.*

Note that the uniform boundedness condition is necessary to prevent concentrations as the energy does not bound any L^s norm for $s > 1$. It is a technical requirement in the structure of the proof, where the key idea is to apply Lions' concentration compactness lemma (Lemma 3.1) to a minimizing sequence, extract a subsequence which is tight in the sense of measures, and use the uniform boundedness to infer weak convergence in L^s for any $1 < s < \infty$. Weak convergence of the minimizing (sub)sequence is sufficient for lower semicontinuity of the energy functionals (Lemma 3.3). We repeat that while the uniform boundedness condition on the density function ρ is a strong assumption, it is supported by results in [36, 35] with $p = 2 - N$, as well as other works that consider power kernels with negative repulsion exponent [4, 5, 60]. The uniform boundedness of the density when minimizing similar energies, is also assumed for example in [2]. As we note in Remark 3.6, this somewhat artificial, albeit justified, assumption should not be necessary, but in our opinion its removal would require substantial additional technical steps.

The radial symmetry assumption on the admissible class when $-N < p < q < 0$ is also a technical assumption. Indeed, in the regime $-N < p < 0 < q$ one can relax this assumption and obtain existence in the general class $\mathcal{A}$. This is possible due to the growth of the kernel at infinity when $q > 0$; however, the argument used to relax the radial symmetry assumption does not apply directly when $-N < p < q < 0$, as the kernel K does not grow indefinitely but approach zero in this regime (see Figure 1(c)).

To our knowledge when $p < 0$ particle simulations in the literature with power-law potentials of the form (1.2) do not reveal any non-radially symmetric steady states (cf. [4, 15, 32]). Also given the isotropic and singular nature of the interaction kernel K it seems reasonable to conjecture that the minimizers of the energy (1.1) defined via power-law potentials are radially symmetric when $p < 0$. Proving that the minimizer is radial, though, is a complicated task and an open problem. Unlike the case of purely attractive kernels (cf. Remark 1.1 and references [6, 22]) where repulsion is given by a diffusion term, symmetrization via Riesz rearrangement techniques [51] do not immediately apply here because of the repulsive-attractive combination in the kernel.

Remark 2.2. As we will show in the next section, the proof of existence of minimizers also applies for potentials of the form

$$K(x) = f(|x|) - \frac{1}{p}|x|^p,$$

where $f(|x|) \to \infty$ as $|x| \to \infty$ and $-N < p < 0$. A similar extension could be made in the case $p > 0$ as well; however, here one needs the function f to grow faster than $|x|^p$ in the long range.

Positive power repulsion $p > 0$. The character of the interaction potential K is very different when $p > 0$. In this regime K does not have a singularity at zero and it allows concentration of densities on sets of dimension less than N. Note that Figure 3(a) shows an associated particle simulation for positive p and q with accumulation along a circle. The results of [4] support this observation via rigorous bounds on the Hausdorff dimension of the support of minimizers. Moreover, for $p > 0$, simulations shows that minimizers need not be radially symmetric (see Figure 3(b)). Therefore we take the energy E defined over probability measures (cf. [4, 5]), that is, in the regime $q > p > 0$, we consider the problem:

$$\text{minimize} \quad E(\mu) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(\frac{1}{q}|x-y|^q - \frac{1}{p}|x-y|^p\right) d\mu(x)d\mu(y), \tag{2.4}$$

over probability measures $\mu \in \mathcal{P}(\mathbb{R}^N)$, endowed with the weak-$*$ topology. Following similar steps as in Theorem 2.1, we prove the existence of a global minimizer.

Theorem 2.3. *There exists a minimum in $\mathcal{P}(\mathbb{R}^N)$ of the problem (2.4) when $q > p > 0$.*

Related to our positive power repulsion case, in [26] the authors consider a class of aggregation equations with interaction potentials which satisfy certain growth and convexity conditions. Using an approach based on the theory of gradient flows they establish existence and uniqueness of global-in-time weak measure solutions. Later in [27], again working with measure solutions, they find sufficient conditions on the interaction potential which guarantee the confinement of localized solutions for all times. In both works their use of a measure theoretic setting as the class of admissible functions enables them a unified analysis of both particle and continuum models.

2.2. **Ground State for Quadratic Attraction and Newtonian Repulsion.** After establishing the existence of global minimizers to the constrained minimization problem (2.1) the next natural question to ask whether one can characterize the global minimizers. In general, this is a very difficult problem to tackle. When $p = 2 - N$, however, the repulsion term corresponds to the Coulomb energy which has a variational characterization via the Newtonian potential. This case was investigated in the context of the evolution equation (1.3) in [36, 35]. There, the authors focused on the existence of symmetric, bounded and compactly-supported steady states and they showed that for any attraction component $q > 2 - N$, a unique such steady state exists. Moreover, numerical experiments suggest that these equilibrium solutions are global attractors for solutions of (1.3).

In particular, for $q = 2$, the steady state considered in [36] consists in a uniform density in a ball. It was shown in [13] that such uniform states (called patch solutions by the authors) are global attractors for the dynamics of (1.3). Also related to this result, in [34] the authors show that in 1D the aggregation equation (1.3) has a unique globally stable steady state when the interaction potential is the sum of Newtonian repulsion and a convex attraction, and the remark that when $q = 2$ this steady state is the characteristic function of an interval. We study these steady states here from a variational point of view, and show that they are global minimizers of (2.1).

Theorem 2.4. *For any $m > 0$ and $M \geqslant \frac{m}{\omega_N}$, the function $\rho(x) = \dfrac{m}{\omega_N}\, \chi_{B(0,1)}(x)$ is the global minimizer of the problem (2.1) in the admissible class $\mathcal{A}$ when $q = 2$, $p = 2 - N$, where ω_N denotes the volume of the unit ball in $\mathbb{R}^N$ and χ denotes the characteristic function of a set.*

Remark 2.5. We note that the case $q = 2$ is rather special. Indeed, since the energy E is translation invariant we can assume, without loss of generality, that the center of mass of admissible densities is zero, that is,

$$\int_{\mathbb{R}^N} x\rho(x)\, dx = 0.$$

A simple calculation leads to

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^2 \rho(x)\rho(y)\, dxdy &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(|x|^2 - 2x\cdot y + |y|^2\right)\rho(x)\rho(y)\, dxdy \\ &= m\int_{\mathbb{R}^N} |x|^2\rho(x)\, dx.\end{aligned}$$

With the attractive term being simplified, the energy (2.1) can be written as

$$E(\rho) = m\int_{\mathbb{R}^N} |x|^2 \rho(x)\, dx - \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^p \rho(x)\rho(y)\, dxdy. \tag{2.5}$$

3. Proofs of the Theorems

In this section we provide proofs of the Theorems 2.1, 2.3 and 2.4.

3.1. **Existence of Global Minimizers. Negative power repulsion** $p < 0$. To prove the existence of a minimizer for (2.1) we will use a direct method from the calculus of variations. The key tool in establishing the existence of minimizers here is the concentration compactness lemma by Lions [52, Lemma I.1].

Lemma 3.1 (Concentration compactness lemma [52])**.** *Let* $\{\rho_n\}_{n\in\mathbb{N}}$ *be a sequence in* $L^1(\mathbb{R}^N)$ *satisfying*

$$\rho_n \geqslant 0 \ \text{in } \mathbb{R}^N, \quad \int_{\mathbb{R}^N} \rho_n(x)\,dx = m,$$

for some fixed $m > 0$*. Then there exists a subsequence* $\{\rho_{n_k}\}_{k\in\mathbb{N}}$ *satisfying one of the three following possibilities:*

(i) *(tightness up to translation) there exists* $y_k \in \mathbb{R}^N$ *such that* $\rho_{n_k}(\cdot + y_k)$ *is tight, that is, for all* $\epsilon > 0$ *there exists* $R > 0$ *such that*

$$\int_{B(y_k,R)} \rho_{n_k}(x)\,dx \geqslant m - \epsilon \qquad \text{for all } k;$$

(ii) *(vanishing)* $\lim_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} \rho_{n_k}(x)\,dx = 0$*, for all* $R > 0$*;*

(iii) *(dichotomy) there exists* $\alpha \in (0, m)$ *such that for all* $\epsilon > 0$*, there exist* $k_0 \geqslant 1$ *and* $\rho_{1,k}$*,* $\rho_{2,k} \in L^1_+(\mathbb{R}^N)$ *satisfying for* $k \geqslant k_0$

$$\|\rho_{n_k} - (\rho_{1,k} + \rho_{2,k})\|_{L^1(\mathbb{R}^N)} \leqslant \epsilon,$$

$$\Big|\|\rho_{1,k}\|_{L^1(\mathbb{R}^N)} - \alpha\Big| \leqslant \epsilon, \quad \Big|\|\rho_{2,k}\|_{L^1(\mathbb{R}^N)} - (m-\alpha)\Big| \leqslant \epsilon,$$

and

$$\operatorname{dist}(\operatorname{supp}(\rho_{1,k}), \operatorname{supp}(\rho_{2,k})) \to \infty \ \text{as } k \to \infty.$$

In certain cases, we will use the following special form of the Hardy–Littlewood–Sobolev inequality to bound the energy from below.

Proposition 3.2 (cf. Theorem 3.1 in [50])**.** *For any* $-N < p < 0$ *and* $f \in L^{2N/(2N+p)}(\mathbb{R}^N)$ *we have*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^p f(x) f(y)\,dxdy \leqslant C(p)\|f\|^2_{L^{2N/(2N+p)}(\mathbb{R}^N)}$$

where the sharp constant $C(p)$ *is given by*

$$C(p) = \pi^{-p/2}\frac{\Gamma(N/2+p/2)}{\Gamma(N+p/2)}\left(\frac{\Gamma(N/2)}{\Gamma(N)}\right)^{-1-p/N}$$

with Γ *denoting the Gamma function.*

Finally, we state and prove a lemma which we will use in establishing the lower semicontinuity of the energy E. A similar argument appears in the proof of Theorem II.1 in [52].

Lemma 3.3. *Let* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ *(or* $\mathcal{A}_r$*) and* $f \in \mathcal{A}$ *(or* $\mathcal{A}_r$*) be given such that* $f_n \rightharpoonup f$ *weakly in* $L^s(\mathbb{R}^N)$ *for some* $1 < s < \infty$*. Then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f_n(x)f_n(y)}{|x-y|^a}\,dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)f(y)}{|x-y|^a}\,dxdy$$

where $0 < a < N$*.*

Proof. First note that

$$\begin{aligned}\left|\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f_n(x)f_n(y)}{|x-y|^a}\,dxdy\right)^{1/2}-\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)f(y)}{|x-y|^a}\,dxdy\right)^{1/2}\right|\\ \leqslant\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f_n(x)-f(x))(f_n(y)-f(y))}{|x-y|^a}\,dxdy\right|^{1/2}.\end{aligned}\tag{3.1}$$

On the other hand, for $R>0$ we have that

$$\begin{aligned}&\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f_n(x)-f(x))(f_n(y)-f(y))}{|x-y|^a}\,dxdy\right|\\ &\leqslant\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f_n(x)-f(x))(f_n(y)-f(y))}{|x-y|^a}\chi_{\{|\cdot|<1/R\}}(|x-y|)\,dxdy\right|\\ &\quad+\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f_n(x)-f(x))(f_n(y)-f(y))}{|x-y|^a}\chi_{\{|\cdot|>R\}}(|x-y|)\,dxdy\right|\\ &\quad+\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f_n(x)-f(x))(f_n(y)-f(y))}{|x-y|^a}\chi_{\{1/R<|\cdot|<R\}}(|x-y|)\,dxdy\right|,\end{aligned}$$

where χ_A denotes the characteristic function of the set A. Since f_n and f are in $\mathcal{A}$ (or $\mathcal{A}_r$), they are uniformly bounded and $\|f_n\|_{L^1(\mathbb{R}^N)}=\|f\|_{L^1(\mathbb{R}^N)}=m$. Hence, the above inequality yields

$$\begin{aligned}\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f_n(x)-f(x))(f_n(y)-f(y))}{|x-y|^a}\,dxdy\right|\leqslant C_1\frac{1}{R^{N-a}}+C_2\frac{1}{R^a}\\ +\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f_n(x)-f(x))(f_n(y)-f(y))}{|x-y|^a}\chi_{\{1/R<|\cdot|<R\}}(|x-y|)\,dxdy\right|,\end{aligned}\tag{3.2}$$

for some constants C_1, $C_2>0$ depending only on a, M and N.

For simplicity of presentation, define

$$g(x-y):=\frac{1}{|x-y|^a}\,\chi_{\{1/R<|\cdot|<R\}}(|x-y|)$$

and note that $g(x-\cdot)\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$. Also, define

$$F_n(x):=\int_{\mathbb{R}^N}f_n(y)g(x-y)\,dy\qquad\text{and}\qquad F(x):=\int_{\mathbb{R}^N}f(y)g(x-y)\,dy.$$

Since $f_n\rightharpoonup f$ weakly in $L^s(\mathbb{R}^N)$, for all $x\in\mathbb{R}^N$ we have that

$$F_n(x)\to F(x),\qquad\text{as } n\to\infty.$$

Moreover, since f, f_n and g are non-negative functions, we have

$$\|f_n*g\|_{L^1(\mathbb{R}^N)}=\|f_n\|_{L^1(\mathbb{R}^N)}\|g\|_{L^1(\mathbb{R}^N)}=\|f\|_{L^1(\mathbb{R}^N)}\|g\|_{L^1(\mathbb{R}^N)}=\|f*g\|_{L^1(\mathbb{R}^N)}$$

which trivially implies that $\|F_n\|_{L^1(\mathbb{R}^N)}\to\|F\|_{L^1(\mathbb{R}^N)}$. Since $|F_n-F|\leqslant|F_n|+|F|$, the function $|F_n|+|F|-|F_n-F|$ is positive. So, applying Fatou's theorem we get that

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N}|F_n|+|F|-|F_n-F|\,dx\geqslant\int_{\mathbb{R}^N}\liminf_{n\to\infty}|F_n|+|F|-|F_n-F|\,dx;$$

hence,

$$2\|F\|_{L^1(\mathbb{R}^N)} - \limsup_{n\to\infty} \int_{\mathbb{R}^N} |F_n - F|\, dx \geqslant 2\|F\|_{L^1(\mathbb{R}^N)}.$$

Thus $\limsup_{n\to 0} \int_{\mathbb{R}^N} |F_n - F|\, dx = 0$, that is, $F_n \to F$ strongly in $L^1(\mathbb{R}^N)$.

Consequently, $F_n \to F$ in $L^s(\mathbb{R}^N)$. This follows from the fact that the strong $L^1(\mathbb{R}^N)$-convergence implies that $F_n(x) \to F(x)$ for a.e. $x \in \mathbb{R}^N$, and this along with the dominated convergence theorem implies the $L^s(\mathbb{R}^N)$-convergence. Now, since $f_n \rightharpoonup f$ weakly in $L^s(\mathbb{R}^N)$, we have that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (f_n(x) - f(x))(f_n(y) - f(y))g(x-y)\, dxdy \to 0$$

as $n \to \infty$. Letting $R \to \infty$ in (3.2) yields, by (3.1), the desired result. □

We now prove the existence theorem for $p < 0$.

Proof of Theorem 2.1. To prove the theorem we consider the two regimes of q separately, as the character of the interaction potential K is quite different in the two cases (see Figures 1(a)-(c)). We minimize the energy over $\mathcal{A}$ when $q > 0$ and over $\mathcal{A}_r$ when $q < 0$.

Case 1: $-N < p < 0 < q$. Let $\{\rho_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ be a minimizing sequence of the problem (2.1), that is, let $\{\rho_n\} \subset \mathcal{A}$ be a sequence such that

$$\lim_{n\to\infty} E(\rho_n) = \inf\{E(\rho)\colon \rho \in \mathcal{A}\}.$$

In this regime both terms of the energy are positive. Hence, $E(\rho) \geqslant 0$ for all $\rho \in \mathcal{A}$, so the above infimum exists and is nonnegative. As $\{\rho_n\}_{n\in\mathbb{N}}$ is a minimizing sequence, for sufficiently large n the energy $E(\rho_n)$ is uniformly bounded.

By the concentration compactness lemma (Lemma 3.1) the sequence $\{\rho_n\}_{n\in\mathbb{N}}$ has a subsequence which satisfies one of the three possibilities: "tightness up to translation", "vanishing" or "dichotomy". We will show that "tightness up to translation" is the only possibility. To this end, suppose "vanishing" occurs. Let $R > 0$ be arbitrary. Then for k large enough

$$\int_{B(0,R)} \rho_{n_k}(x)\, dx \leqslant \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} \rho_{n_k}(x)\, dx < m/2.$$

Since $\rho_{n_k} \in \mathcal{A}$ this implies that

$$\int_{\mathbb{R}^N\setminus B(0,R)} \rho_{n_k}(x)\, dx \geqslant m/2 > 0. \tag{3.3}$$

Now we are going to use the fact that the attractive term grows indefinitely at infinity. Since ρ_{n_k} are positive, by (3.3) we have that

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_{n_k}(x)\rho_{n_k}(y)\, dxdy &\geqslant \int_{\mathbb{R}^N}\int_{|x-y|>R} |x-y|^q \rho_{n_k}(x)\rho_{n_k}(y)\, dxdy \\ &\geqslant R^q \int_{\mathbb{R}^N} \rho_{n_k}(x) \left(\int_{|x-y|>R} \rho_{n_k}(y)\, dy\right) dx \\ &\geqslant R^q\, m(m/2).\end{aligned}$$

Thus
$$E(\rho_{n_k}) \geqslant \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_{n_k}(x)\rho_{n_k}(y)\,dxdy \geqslant C\,R^q.$$
As $q>0$, for sufficiently large $R>0$ there exists a sufficiently large $k_0>0$ such that for all $n_k>k_0$, $E(\rho_{n_k}) \geqslant CR^q > \inf\{E(\rho)\colon \rho\in\mathcal{A}\}$, contradicting the fact that ρ_{n_k} is a minimizing sequence. Therefore "vanishing" does not occur.

Next, suppose "dichotomy" occurs. Using the notation of Lemma 3.1(iii), let
$$d_k := \operatorname{dist}(\operatorname{supp}(\rho_{1,k}), \operatorname{supp}(\rho_{2,k}))$$
denote the distance between the supports of $\rho_{1,k}$ and $\rho_{2,k}$. We can further assume that the supports of $\rho_{1,k}$ and $\rho_{2,k}$ are disjoint.

Inspecting again the attraction term we get that for some constant $C>0$,
$$\begin{aligned}\frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} &|x-y|^q \rho_{n_k}(x)\rho_{n_k}(y)\,dxdy \\ &\geqslant \frac{C}{q}\int_{\operatorname{supp}(\rho_{1,k})}\int_{\operatorname{supp}(\rho_{2,k})} |x-y|^q \rho_{1,k}(x)\rho_{2,k}(y)\,dxdy \\ &\geqslant \frac{C}{q}\, d_k^q\, \|\rho_{1,k}\|_{L^1(\mathbb{R}^N)}\|\rho_{2,k}\|_{L^1(\mathbb{R}^N)}.\end{aligned}$$
Since $d_k\to\infty$ as $k\to\infty$, and $\|\rho_{i,k}\|_{L^1(\mathbb{R}^N)}$ does not converge to zero, the above estimate gives that $E(\rho_{n_k})\to\infty$, contradicting again the fact that ρ_{n_k} is a minimizing sequence. Thus "dichotomy" does not occur.

Therefore "tightness up to translation" is the only possibility, i.e., there exists a sequence $\{y_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}^N$ such that
$$\text{for all } \epsilon>0 \text{ there exists } R>0 \text{ satisfying } m \geqslant \int_{B(y_k,R)} \rho_{n_k}(x)\,dx \geqslant m-\epsilon. \tag{3.4}$$

Now, let $\overline{\rho}_{n_k}(x) = \rho_{n_k}(x+y_k)$ and note that $E(\rho_{n_k}) = E(\overline{\rho}_{n_k})$ by translation invariance of the energy E. Thus, $\{\overline{\rho}_{n_k}\}_{k\in\mathbb{N}}$ is also a minimizing sequence. Since $\{\rho_{n_k}\}_{k\in\mathbb{N}}\subset\mathcal{A}$, all members of the sequence are uniformly bounded in $L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ and passing to a subsequence if necessary, we may assume that
$$\overline{\rho}_{n_k} \rightharpoonup \rho_0 \text{ weakly in } L^s(\mathbb{R}^N)$$
for some $1<s<\infty$ and some $\rho_0\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ [2]. Moreover, by (3.4),
$$\int_{\mathbb{R}^N} \rho_0(x)\,dx = m,$$
or in other words, when passing to the limit as $k\to\infty$ the sequence $\overline{\rho}_{n_k}$ does not "leak-out" at infinity. To show that $\rho_0\geqslant 0$ a.e. let
$$S := \{x\in\mathbb{R}^N\colon \rho_0(x)<0\}.$$

[2] In fact, the sequence $\overline{\rho}_{n_k}$ converges weakly to ρ_0 in $L^s(\mathbb{R}^N)$ for every $1<s<\infty$ because of the uniform bound on the sequence. The weak convergence holds for $s=1$, as well, by (3.4) and since the translation sequence $\{y_k\}_{k\in\mathbb{N}}$ can be taken to be zero by the translation invariance of the energy.

Then the characteristic function of S, χ_S, is an admissible test function for the weak convergence of $\overline{\rho}_{n_k}$, so we get that

$$\int_S \overline{\rho}_{n_k}(x)\,dx \to \int_S \rho_0(x)\,dx < 0.$$

However, since $\overline{\rho}_{n_k} \in \mathcal{A}$, we see that

$$\liminf_{k\to\infty} \int_S \overline{\rho}_{n_k}(x)\,dx \geqslant 0;$$

hence, S has measure zero. Similarly we can show that $\|\rho_0\|_{L^\infty(\mathbb{R}^N)} \leqslant M$. Thus $\rho_0 \in \mathcal{A}$.

Next we need to show that the energy is weakly lower semicontinuous. Here, with an abuse of notation, we will drop the bar on $\overline{\rho}_n$, and simply denote them by ρ_n.

By Lemma 3.3, the repulsive part is weakly lower semicontinuous and we have that

$$-\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^p \rho_n(x)\rho_n(y)\,dxdy \to -\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^p \rho_0(x)\rho_0(y)\,dxdy \tag{3.5}$$

as $n \to \infty$.

On the other hand, for the attractive part, define

$$G_n(x) = \int_{B(0,R)} |x-y|^q \rho_n(y)\,dy \quad \text{and} \quad G_0(x) = \int_{B(0,R)} |x-y|^q \rho_0(y)\,dy,$$

for any fixed $R > 0$. Note that since $\|\rho_0\|_{L^\infty(\mathbb{R}^N)} \leqslant M$ and $q > 0$, we see that $G_0 \in L^\infty(B(0,R))$, in particular, $G_0 \in L^{s/(s-1)}(B(0,R))$. Therefore, by the weak convergence of ρ_n in $L^s(B(0,R))$,

$$\int_{B(0,R)} G_0(x)[\rho_n(x) - \rho_0(x)]\,dx \to 0 \tag{3.6}$$

as $n \to \infty$. Also, since ρ_n are uniformly bounded, taking $\int_{B(0,R)} |\cdot - y|^q\,dy \in L^{s/(s-1)}(B(0,R))$ as a test function, we see that

$$\int_{B(0,R)} \rho_n(x)[G_n(x) - G_0(x)]\,dx \to 0 \tag{3.7}$$

as $n \to \infty$, by the weak convergence of ρ_n in $L^s(B(0,R))$.

Thus, using (3.6) and (3.7), we have that

$$\begin{aligned}\int_{B(0,R)} G_n(x)\rho_n(x)\,dx = &\int_{B(0,R)} G_0(x)[\rho_n(x) - \rho_0(x)]\,dx \\ &+ \int_{B(0,R)} \rho_n(x)[G_n(x) - G_0(x)]\,dx + \int_{B(0,R)} G_0(x)\rho_0(x)\,dx\end{aligned}$$

converges to

$$\int_{B(0,R)} G_0(x)\rho_0(x)\,dx$$

as $n \to \infty$. Hence,

$$\liminf_{n\to\infty} \int_{B(0,R)}\int_{B(0,R)} |x-y|^q \rho_n(x)\rho_n(y)\,dxdy = \int_{B(0,R)}\int_{B(0,R)} |x-y|^q \rho_0(x)\rho_0(y)\,dxdy.$$

Now, by (3.4), for given $\epsilon > 0$, we can choose $R > 0$ such that

$$\int_{B(0,R)} \rho_0(x)\, dx \geqslant m - \epsilon.$$

Then, for such R, since $E(\rho_0) < \infty$, we can control the excess of the attractive part on $\mathbb{R}^N \setminus B(0,R)$ and we get that

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_0(x)\rho_0(y)\, dxdy &\leqslant \int_{B(0,R)}\int_{B(0,R)} |x-y|^q \rho_0(x)\rho_0(y)\, dxdy + C\epsilon \\ &\leqslant \liminf_{n\to\infty} \int_{B(0,R)}\int_{B(0,R)} |x-y|^q \rho_n(x)\rho_n(y)\, dxdy + C\epsilon \quad (3.8) \\ &\leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_n(x)\rho_n(y)\, dxdy + C\epsilon.\end{aligned}$$

Letting $\epsilon \to 0$ and combining with (3.5) yields

$$\inf\{E(\rho) : \rho \in \mathcal{A}\} \leqslant E(\rho_0) \leqslant \liminf_{n\to\infty} E(\rho_n) = \inf\{E(\rho) : \rho \in \mathcal{A}\},$$

that is, ρ_0 is a solution to the minimization problem (2.1) in the regime $-N < p < 0 < q$.

Case 2: $-N < p < q < 0$. In this regime, the character of the interaction potential is quite different than in the previous case. Now the attractive term is strictly negative whereas the repulsive part of the energy E is still strictly positive. We also remind the reader that in this regime we prove the existence of a global minimizer of the energy under the additional assumption of radial symmetry on the admissible functions, i.e., in the class $\mathcal{A}_r$. Here we use the radial symmetry to rule out "dichotomy" and we believe that radial symmetry assumption can be relaxed in this regime, as well; however, the argument used to rule out "vanishing" of a minimizing sequence when $q > 0$ does not apply directly in this case as the kernel K does not grow indefinitely but approach zero in this regime (see Figure 1(c)).

First, using Proposition 3.2 we see that the attractive term is bounded below, and we conclude that in this regime

$$\inf\{E(\rho)\colon \rho \in \mathcal{A}_r\} > -\infty.$$

Next, looking at the scaling

$$\rho_\lambda(x) = \frac{1}{\lambda^N}\rho\left(\frac{x}{\lambda}\right)$$

we see that $\rho_\lambda \in \mathcal{A}_r$ for $\lambda \geqslant 1$, and the energy of ρ_λ is given by

$$E(\rho_\lambda) = \frac{\lambda^q}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho(x)\rho(y)\, dxdy - \frac{\lambda^p}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^p \rho(x)\rho(y)\, dxdy,$$

for any given function $\rho \in \mathcal{A}_r$. Note that, in particular, we can choose ρ to be the characteristic function of a ball. Since $-N < p < q < 0$, for λ large enough we get that $E(\rho_\lambda) < 0$, and hence,

$$I_m := \inf\{E(\rho)\colon \rho \in \mathcal{A}_r\} < 0.$$

Again, we will make use of the concentration compactness lemma, Lemma 3.4, and show that for a minimizing sequence ρ_n the possibilities of "vanishing" and "dichotomy" do not occur.

Suppose "vanishing" occurs. Since $I_m < 0$ in this regime and since the repulsive part is strictly positive, looking at the attractive part we have that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_n(x)\rho_n(y)\,dxdy > 0. \tag{3.9}$$

Let $R > 1$ and $q = -a$ for $0 < a < N$. Then, as in the proof of Lemma 3.3,

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_n(x)\rho_n(y)\,dxdy &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_n(x)\rho_n(y)}{|x-y|^a}\chi_{\{|x|<1/R\}}(|x-y|)\,dxdy \\ &\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_n(x)\rho_n(y)}{|x-y|^a}\chi_{\{1/R<|x|<R\}}(|x-y|)\,dxdy \\ &\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_n(x)\rho_n(y)}{|x-y|^a}\chi_{\{|x|>R\}}(|x-y|)\,dxdy \\ &\leqslant \frac{CmM}{R^{N-a}} + \frac{m^2}{R^a} + R^a \int_{\mathbb{R}^N} \rho_n(x) \int_{|x-y|<R} \rho_n(y)\,dydx \\ &\leqslant \frac{CmM}{R^{N-a}} + \frac{m^2}{R^a} + R^a\, m \sup_{x\in\mathbb{R}^N} \left(\int_{|x-y|<R} \rho_n(y)\,dy \right)\end{aligned}$$

where C is positive constant depending only on a and N, M is the uniform bound on ρ_n and m is the mass of ρ_n, as before.

Since ρ_n vanishes by Lemma 3.1 (ii), we get that as $n \to \infty$ the last term in the above inequality is zero; hence,

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_n(x)\rho_n(y)\,dxdy \leqslant \frac{CmM}{R^{N-a}} + \frac{m^2}{R^a}.$$

Letting $R \to \infty$, since $0 < a < N$, this yields that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_n(x)\rho_n(y)\,dxdy \leqslant 0,$$

contradicting (3.9). Thus "vanishing" does not occur.

To show that "dichotomy" does not occur, first we need to prove a subadditivity condition similar to the one in [52]. As in [6, Lemma 1], we can prove a weaker subadditivity condition which states that

$$\text{for } m_1 > m_2 \text{ we have } I_{m_1} < I_{m_2}, \tag{3.10}$$

where, as above, I_{m_i} denotes the infimum of E over $\mathcal{A}_r^{m_i}$ with mass constraint $\int_{\mathbb{R}^N} \rho(x)\,dx = m_i$. Here we choose to display the dependence of the admissible class $\mathcal{A}_r$ on the mass by using the notation $\mathcal{A}_r^{m_i}$ to avoid confusion.

Suppose $m_1 > m_2$ and let $\psi \in \mathcal{A}_r^{m_2}$ be an arbitrary function. For

$$c := \frac{m_2}{m_1} < 1$$

define $\rho \in \mathcal{A}_r^{m_1}$ such that

$$\psi = c\,\rho.$$

Then we have that

$$E[\psi] = c^2\, E[\rho].$$

Note that since $I_{m_1} < 0$ in this regime and since $c^2 < 1$ we have that

$$c^2\, I_{m_1} > I_{m_1}.$$

Thus

$$E[\psi] = c^2\, E[\rho] \geqslant c^2\, I_{m_1} > I_{m_1},$$

and taking the infimum of both sides over $\mathcal{A}_r^{m_2}$ implies that

$$I_{m_2} > I_{m_1}.$$

Now, suppose "dichotomy" occurs, that is, there exists $\alpha \in (0,m)$ such that for all $\epsilon > 0$, there exist $k_0 \geqslant 1$ and $\rho_{1,k},\ \rho_{2,k} \in L^1_+(\mathbb{R}^N)$ satisfying for $k \geqslant k_0$

$$\|\rho_{n_k} - (\rho_{1,k} + \rho_{2,k})\|_{L^1(\mathbb{R}^N)} \leqslant \epsilon,$$
$$\Big|\|\rho_{1,k}\|_{L^1(\mathbb{R}^N)} - \alpha\Big| \leqslant \epsilon, \quad \Big|\|\rho_{2,k}\|_{L^1(\mathbb{R}^N)} - (m-\alpha)\Big| \leqslant \epsilon,$$

and

$$\operatorname{dist}(\operatorname{supp}(\rho_{1,k}), \operatorname{supp}(\rho_{2,k})) \to \infty \text{ as } k \to \infty.$$

Furthermore, after defining $v_k := \rho_{n_k} - (\rho_{1,k} + \rho_{2,k})$ we can assume that

$$0 \leqslant \rho_{1,k}, \rho_{2,k}, v_k \leqslant \rho_{n_k} \text{ and } \rho_{1,k}\rho_{2,k} = \rho_{1,k}v_k = \rho_{2,k}v_k = 0 \text{ a.e.} \tag{3.11}$$

We have that, for any $0 < a < N$,

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_{n_k}(x)\rho_{n_k}(y)}{|x-y|^a}\,dxdy &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_{1,k}(x)\rho_{1,k}(y)}{|x-y|^a}\,dxdy \\ &+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_{2,k}(x)\rho_{2,k}(y)}{|x-y|^a}\,dxdy + 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_{1,k}(x)\rho_{2,k}(y)}{|x-y|^a}\,dxdy \\ &+ 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_{n_k}(x)v_k(y)}{|x-y|^a}\,dxdy - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v_k(x)v_k(y)}{|x-y|^a}\,dxdy.\end{aligned} \tag{3.12}$$

The last two terms above vanish as $k \to \infty$ using the integrability of the kernel around zero, the uniform bound on ρ_{n_k} and the fact that $\|v_k\|_{L^1(\mathbb{R}^N)} \to 0$. Since $\operatorname{dist}(\operatorname{supp}(\rho_{1,k}), \operatorname{supp}(\rho_{2,k})) \to \infty$ as $k \to \infty$, and $\lim_{|x|\to\infty} K(|x|) = 0$ in this regime, the third term on the right hand side of (3.12) goes to zero as $k \to \infty$.

Again, since $\operatorname{dist}(\operatorname{supp}(\rho_{1,k}), \operatorname{supp}(\rho_{2,k})) \to \infty$ as $k \to \infty$, we see that one of the components of ρ_n is localized and the other component (say $\rho_{2,k}$, without loss of generality) spreads to infinity, i.e., $\operatorname{dist}(0, \operatorname{supp}(\rho_{2,k})) \to \infty$ as $k \to \infty$. Also, as the supports of $\rho_{1,k}$, $\rho_{2,k}$ and v_k are disjoint as noted in (3.11) and the functions ρ_{n_k} are radially symmetric, so are the functions $\rho_{2,k}$. Using the radial symmetry of $\rho_{2,k}$, and recalling that the kernel $K(|x|)$ approaches zero as $|x| \to \infty$, we get that

$$\lim_{k\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\rho_{2,k}(x)\rho_{2,k}(y)\,dxdy = 0.$$

This follows from the fact that the radial symmetry of the *functions* $\rho_{2,k}$ guarantee that their supports do not extend to infinity in one direction but rather in such a way that one can bound the energy from below by

$$\int_S\int_{-S} K(x-y)\rho_{2,k}(x)\rho_{2,k}(y)\,dxdy,$$

where S is a sector defined by $x \in \mathbb{R}^N$ with $|x| > R$ and such that the angle of the vector x with a fixed direction v is less than a constant angle, say, $\pi/4$.

These observations combined with (3.12) imply that

$$\begin{aligned} I_m &= \lim_{k\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\rho_{n_k}(x)\rho_{n_k}(y)\,dxdy \\ &\geqslant \liminf_{k\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\rho_{1,k}(x)\rho_{1,k}(y)\,dxdy \\ &= I_\alpha, \end{aligned} \tag{3.13}$$

which contradicts the weak subadditivity condition (3.10). Thus "dichotomy" does not occur.

As in the first case, "tightness up to translation" is the only possibility. Therefore the weak limit ρ_0 of the translated sequence $\overline{\rho}_n$ satisfies the mass constraint and hence, is a member of $\mathcal{A}_r$.

The weak lower semicontinuity in this regime follows directly from Lemma 3.3 as both attractive and repulsive terms of the energy are of the form considered in the lemma and by (3.4) the assumptions of the lemma are satisfied. We conclude that the minimization problem (2.1) has a solution when $-N < p < q < 0$. □

Remark 3.4. The concentration compactness principle suffices to establish a weaker form compactness so that we can pass to a weak limit in the sequence ρ_n. However, the sequence does not necessarily convergence strongly to ρ in any $L^s(\mathbb{R}^N)$. Indeed, strong convergence can fail due to mass leaking out at infinity and/or because of oscillations. By the tightness of the sequence $\{\rho_n\}_{n\in\mathbb{N}}$ the former does not happen; but, we cannot rule out the oscillations of ρ_n.

On the other hand, we note that for functionals which contain a term that is convex in ρ (cf. Remark 1.1), one can further show that the convergence of $\{\rho_n\}_{n\in\mathbb{N}}$ is strong (cf. [6, 52]).

Remark 3.5. When $N = 2$ the Newtonian potential is given by $-\frac{1}{2\pi}\log|x|$. Either considering the logarithmic term as the repulsion in

$$K(x) = \frac{1}{q}|x|^q - \log|x|, \qquad q > 0, \quad x \in \mathbb{R}^2$$

or as the attraction in

$$K(x) = \log|x| - \frac{1}{p}|x|^p, \qquad -2 < p < 0, \quad x \in \mathbb{R}^2$$

the proof of Theorem 2.1 applies since the properties of the interaction potential (singularity at the origin and blow-up at infinity) remain the same as in higher dimensions.

Remark 3.6 (Uniform Boundedness). The uniform L^∞ bound is used for admissible densities ρ in the proof of Theorem 2.1 to control the energy near the singularity of K, and provide weak compactness in some function space. We have motivated, for example from the point of view of gradient flow dynamics, why this assumption is natural, or more precisely, acceptable. However, we do not believe it is essential but rather convenient for our proof. In fact, we believe that the assumption can be relaxed by just taking densities in some L^s (not necessarily uniformly bounded), using tightness of a minimizing sequence to imply convergence of measures, and then showing that, due to the negative-power repulsion, finite energy rules out concentrations (i.e. the densities remain *functions*). During the review process of this work we became aware of the work [25] where the authors address this issue when $-N < p < 2-N$ using an obstacle problem interpretation.

Positive power repulsion $p > 0$. The main tool in establishing the existence of minimizers for (2.4) is, again, the concentration compactness principle. We refer to [57, Section 4.3] for the following lemma.

Lemma 3.7 (Concentration compactness lemma for measures)**.** *Let* $\{\mu_n\}_{n\in\mathbb{N}}$ *be a sequence of probability measures on* $\mathbb{R}^N$*. Then there exists a subsequence* $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ *satisfying one of the three following possibilities:*

(i) *(tightness up to translation) there exists* $y_k \in \mathbb{R}^N$ *such that for all* $\epsilon > 0$ *there exists* $R > 0$ *with the property that*
$$\int_{B(y_k,R)} d\mu_{n_k}(x) \geqslant 1-\epsilon \qquad \text{for all } k.$$

(ii) *(vanishing)* $\displaystyle\lim_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} d\mu_{n_k}(x) = 0$*, for all* $R > 0$*;*

(iii) *(dichotomy) there exists* $\alpha \in (0,1)$ *such that for all* $\epsilon > 0$*, there exists a number* $R > 0$ *and a sequence* $\{y_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^N$ *with the following property:*

Given $R' > R$ *there are non-negative measures* μ_k^1 *and* μ_k^2 *such that*
$$0 \leqslant \mu_k^1 + \mu_k^2 \leqslant \mu_{n_k}\,,$$
$$\operatorname{supp}(\mu_k^1) \subset B(y_k,R),\ \operatorname{supp}(\mu_k^2) \subset \mathbb{R}^N \setminus B(y_k,R')\,,$$
$$\limsup_{k\to\infty}\left(\left|\alpha - \int_{\mathbb{R}^N} d\mu_k^1(x)\right| + \left|(1-\alpha) - \int_{\mathbb{R}^N} d\mu_k^2(x)\right|\right) \leqslant \epsilon.$$

Proof of Theorem 2.3. For any $\mu \in \mathcal{P}(\mathbb{R}^N)$ we have that $\int_{\mathbb{R}^N} d\mu(x) = 1$. Also when $q > p > 0$ the interaction potential satisfies $K(|x|) \geqslant 1/q - 1/p$. Thus
$$\inf\{E(\mu)\colon \mu \in \mathcal{P}(\mathbb{R}^N)\} > -\infty.$$
Since $K(|x|) \leqslant 0$ when $0 \leqslant |x| \leqslant (q/p)^{1/(q-p)}$ we see that the above infimum is negative.

Now let $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}^N)$ be a minimizing sequence of the problem (2.4). Then by the concentration compactness lemma for measures there is a subsequence of $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ which satisfies one of the three possibilities in Lemma 3.7.

Suppose "vanishing" occurs, i.e., for $0 < \epsilon < 1$ and $R > 0$ and for k sufficiently large enough we have that
$$\int_{B(y,R)} d\mu_{n_k}(x) < \epsilon$$
for any $y \in \mathbb{R}^N$. This implies that
$$\int_{\mathbb{R}^N\setminus B(0,R)} d\mu_{n_k}(x) \geqslant 1-\epsilon > 0.$$

Note that since K is a polynomial of $|x|$ there exists a constant $C_{p,q} > 0$ depending on p and q only such that

$$K(x) \geqslant |x|^{q-p} - C_{p,q} \tag{3.14}$$

with $q - p > 0$.

Now looking at the energy and using the indefinite growth of the interaction potential K as $|x| \to \infty$ (see Figure 1(d)) we see that

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\, d\mu_{n_k}(x) d\mu_{n_k}(y) &\geqslant \int_{\mathbb{R}^N}\int_{|x-y|>R} \left(|x-y|^{q-p} - C_{p,q}\right)\, d\mu_{n_k}(x) d\mu_{n_k}(y)\\ &\geqslant R^{q-p}\int_{\mathbb{R}^N}\left(\int_{|x-y|>R} d\mu_{n_k}(y)\right) d\mu_{n_k}(x) - C_{p,q}\\ &\geqslant R^{q-p}(1-\epsilon) - C_{p,q};\end{aligned}$$

hence, for sufficiently large R, there exists $k_0 > 0$ such that the energy $E[\mu_{n_k}] > 0$ for all $n_k > k_0$. This contradicts the fact that $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ is a minimizing sequence.

Similarly, if we assume that "dichotomy" occurs, looking at the energy and using (3.14) we get that

$$\begin{aligned}\liminf_{k\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\, d\mu_{n_k}(x) d\mu_{n_k}(y) &\geqslant \liminf_{k\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\, d\mu_k^1(x) d\mu_k^2(y)\\ &\geqslant (R'-R)^{q-p}\,\alpha(1-\alpha) - C_{p,q}.\end{aligned}$$

Again, since $q > p > 0$ and $K(|x|) \nearrow \infty$ as $|x| \to \infty$, by taking R' large enough we get that

$$\liminf_{k\to\infty} E[\mu_{n_k}] > 0,$$

a contradiction.

Therefore "tightness up to translation" is the only possibility. As in the case of Theorem 2.1 for $q > 0$, the centers y_k associated with the translation can be taken to be zero by the translation invariance of the energy. Hence we may assume the sequence of probability measures, $\{\mu_n\}_{n\in\mathbb{N}}$ is tight. Then, by the Prokhorov's theorem (cf. [16, Theorem 4.1]) there exists a further subsequence of $\{\mu_n\}_{n\in\mathbb{N}}$ which we still index by n, and a measure $\mu_0 \in \mathcal{P}(\mathbb{R}^N)$ such that

$$\mu_n \overset{\text{weak}-*}{\rightharpoonup} \mu_0$$

in $\mathcal{P}(\mathbb{R}^N)$ as $n \to \infty$.

To show weak lower semi-continuity of $E(\mu)$ we will proceed as in the proof of Theorem 2.1, paying attention to the fact that in this regime K becomes negative.

Since the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is tight, for any given $\epsilon > 0$ there exists $r > 0$ such that

$$\int_{B(0,r)} d\mu_0(x) \geqslant 1 - \epsilon.$$

Choose $R := \max\{(q/p)^{1/(q-p)} + 1, r\}$, and define

$$\tilde{G}_n(x) := \int_{B(0,R)} K(x,y)\, d\mu_n(y) \quad \text{and} \quad \tilde{G}_0(x) := \int_{B(0,R)} K(x,y)\, d\mu_0(y).$$

As $K(x,y)$ is continuous in x on $B(0,R)$, the sequence of functions $\tilde{G}_n$ converges uniformly to $\tilde{G}$ on $C(\overline{B(0,R)})$ by the Arzela–Ascoli theorem, using the compactness of the closed ball and the equicontinuity of $\tilde{G}_n$. Then, by the uniform convergence of $\tilde{G}_n$ and the weak-$*$ convergence of μ_n we get that

$$\liminf_{n\to\infty}\int_{B(0,R)}\int_{B(0,R)} K(x,y)\,d\mu_n(x)d\mu_n(y) = \int_{B(0,R)}\int_{B(0,R)} K(x,y)\,d\mu_0(x)d\mu_0(y).$$

Since $E(\mu_0)<\infty$, again, the energy on $\mathbb{R}^N\setminus B(0,R)$ is controlled and the above equality, as in (3.8), yields

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x,y)\,d\mu_0(x)d\mu_0(y) &\leqslant \int_{B(0,R)}\int_{B(0,R)} K(x,y)\,d\mu_0(x)d\mu_0(y) + C\epsilon\\ &\leqslant \liminf_{n\to\infty}\int_{B(0,R)}\int_{B(0,R)} K(x,y)\,d\mu_n(x)d\mu_n(y) + C\epsilon\\ &\leqslant \liminf_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x,y)\,d\mu_n(x)d\mu_n(y) + C\epsilon.\end{aligned}$$

Sending ϵ to 0 gives the weak lower semi-continuity of E; hence, $\mu_0\in\mathcal{P}(\mathbb{R}^N)$ is a solution of the minimization problem (2.4). □

3.2. Ground State for Quadratic Attraction and Newtonian Repulsion.

Finally, going back to the setting of admissible *functions*, i.e., working in $\mathcal{A}$ defined by (2.2), we will prove Theorem 2.4 and characterize the ground state when $q=2$ and $p=2-N$. To establish this we need to derive the full Euler–Lagrange equations for the energy E. We obtain these equations not in the restricted class of radially symmetric densities but in the wider class (2.2). The same Euler–Lagrange equations were formally obtained in [10] in one dimension and derived in the context of minimization with respect to the 2-Wasserstein distance in [4]. Similar conditions were considered in [10, Section 2.3]; however, here we take a more direct and elementary approach in the spirit of a variational inequality.

Lemma 3.8 (First Variation of the Energy)**.** *Let $\rho_0\in\mathcal{A}$ be a minimizer of the energy E. Then we have*

$$\begin{aligned}\Lambda(x)&\geqslant\eta \quad \text{a.e. on the set} \quad \{x:\rho_0(x)=0\},\\ \Lambda(x)&=\eta \quad \text{a.e. on the set} \quad \{x:\rho_0(x)>0\},\end{aligned} \tag{3.15}$$

where

$$\Lambda(x) := 2\int_{\mathbb{R}^N}\left(\frac{1}{q}|x-y|^q - \frac{1}{p}|x-y|^p\right)\rho_0(y)\,dy, \tag{3.16}$$

and η is a constant.

Proof. Proceeding as in [51], let ρ_0 be a minimizer of E and let $\zeta\in Z$ be an arbitrary function. For $0\leqslant\epsilon\leqslant 1$, consider

$$\rho_\epsilon(x) := \rho_0(x) + \epsilon\left(\zeta(x) - \frac{\int_{\mathbb{R}^N}\zeta(x)\,dx}{m}\rho_0(x)\right). \tag{3.17}$$

Clearly $\int_{\mathbb{R}^N}\rho_\epsilon(x)\,dx = m$. Also, since $\zeta\in Z$, we have $\rho_\epsilon\geqslant 0$.

Note that the function

$$e(\epsilon) := E\left[\rho_0(x) + \epsilon\left(\zeta(x) - \frac{\int_{\mathbb{R}^N}\zeta(x)\,dx}{m}\rho_0(x)\right)\right] \tag{3.18}$$

is defined on the interval $[0,1]$. Indeed, 0 is a boundary point, since for any $\epsilon < 0$, the perturbation function ρ_ϵ can be made negative; hence, it is not a member of the admissible class.

Now the minimality of ρ_0 implies that

$$e'(0^+) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0^+} E(\rho_\epsilon) \geqslant 0. \tag{3.19}$$

In explicit terms this means that

$$\begin{aligned}\frac{d}{d\epsilon}\bigg|_{\epsilon=0^+} E(\rho_\epsilon) = 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} &\left(\frac{1}{q}|x-y|^q - \frac{1}{p}|x-y|^p\right)\rho_0(y)\zeta(x)\,dxdy \\ &- 2\left(\frac{\int_{\mathbb{R}^N}\zeta(x)\,dx}{m}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(\frac{1}{q}|x-y|^q - \frac{1}{p}|x-y|^p\right)\rho_0(x)\rho_0(y)\,dxdy \\ &= \int_{\mathbb{R}^N}\Lambda(x)\zeta(x)\,dx - \eta\int_{\mathbb{R}^N}\zeta(x)\,dx \\ &= \int_{\mathbb{R}^N}(\Lambda(x)-\eta)\zeta(x)\,dx,\end{aligned}$$

where Λ was defined in (3.16), and

$$\eta := \frac{\int_{\mathbb{R}^N}\Lambda(x)\rho_0(x)\,dx}{m}. \tag{3.20}$$

Hence, we get that

$$\int_{\mathbb{R}^N}(\Lambda(x)-\eta)\zeta(x)\,dx \geqslant 0. \tag{3.21}$$

The inequality (3.21) above holds, in particular, for all nonnegative functions $\zeta \in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N}\zeta(x)\,dx \leqslant \frac{m}{2}.$$

This, in turn, implies that

$$\Lambda(x) - \eta \geqslant 0 \quad \text{a.e.}$$

Moreover, note that η is the average of Λ with respect to the measure $\rho_0(x)dx$, and hence the condition $\Lambda(x) \geqslant \eta$ a.e. implies that $\Lambda(x) = \eta$ for a.e. x where $\rho_0(x) > 0$. This establishes (3.15). □

It is evident that in Lemma 3.8, we actually do not need ρ_0 to be a minimizer but simply a critical point in the sense of (3.19) holding for all $\zeta \in Z$.

Remark 3.9 (Equipartition of the Energy)**.** Note that the equipartition (up to constants) of the energy is a necessary condition for criticality. Namely, when for $\rho \in \mathcal{A}$, and for any $\lambda \neq 0$ we consider the rescaled function ρ^λ given by $\rho^\lambda(x) := (1/\lambda^N)\rho(x/\lambda)$, then a necessary condition for criticality (in particular for being a local minimizer) is that

$$\frac{d}{d\lambda}\bigg|_{\lambda=1} E(\rho^\lambda) = 0. \tag{3.22}$$

In particular, if $\rho_0 \in \mathcal{A}$ satisfies (3.22), then

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|x-y|^q\rho_0(x)\rho_0(y)\,dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|x-y|^p\rho_0(x)\rho_0(y)\,dxdy \tag{3.23}$$

Now we can prove Theorem 2.4.

Proof of Theorem 2.4. The existence of a minimizer was established in Theorem 2.1. Now, first we note that since $M \geqslant \frac{m}{\omega_N}$, the function ρ is in the admissible class $\mathcal{A}$, where $\omega_N = \frac{\pi^{N/2}}{\Gamma(N/2+1)}$ denotes the volume of the unit ball in $\mathbb{R}^N$. Next, we check that ρ is a critical point of the functional E, i.e., that ρ satisfies (3.15). As noted in Remark 2.5, the attractive term of the energy simplifies when $q = 2$. On the other hand, when $p = 2 - N$ the repulsive part is the Newtonian potential (i.e., $-\Delta_y(|x-y|^{2-N}) = N(N-2)\omega_N \delta_x$), and

$$\Phi(x) := \int_{B(0,1)} \frac{1}{N(N-2)\omega_N |x-y|^{N-2}}\, dy$$

solves the Poisson problem

$$-\Delta\Phi(x) = \begin{cases} 1 & \text{if } |x| \leqslant 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

Since the right-hand side of the Poisson problem is radial, so is $\Phi(x)$. We use the expression of the Laplacian on $\mathbb{R}^N$ in hyper-spherical coordinates,

$$-\Delta\Phi(x) = -\frac{1}{r^{N-1}} \frac{d}{dr}\left(r^{N-1} \frac{d\Phi(r)}{dr} \right),$$

with $r = |x|$, and integrate once to get

$$\frac{d\Phi(r)}{dr} = \begin{cases} -\frac{r}{N} & \text{if } r \leqslant 1, \\ -\frac{1}{Nr^{N-1}} & \text{if } r > 1. \end{cases}$$

Integrating one more time and using the fact that $\Phi \in C^1$ by elliptic regularity, we get that

$$\Phi(r) = \begin{cases} -\frac{r^2}{2N} + \frac{1}{2(N-2)} & \text{if } r \leqslant 1, \\ \frac{1}{N(N-2)r^{N-2}} & \text{if } r > 1. \end{cases}$$

Then we calculate the function $\Lambda(x)$ given by (3.16) to find

$$\begin{aligned} \Lambda(x) &= 2 \int_{\mathbb{R}^N} \left(\frac{1}{2}|x-y|^2 - \frac{1}{2-N}|x-y|^{2-N} \right) \frac{m}{\omega_N} \chi_{B(0,1)}(y)\, dy \\ &= \frac{m}{\omega_N} \left(\int_{B(0,1)} |x|^2 + |y|^2\, dy \right) + \frac{2m}{\omega_N(N-2)} \int_{B(0,1)} \frac{1}{|x-y|^{N-2}}\, dy \\ &= \begin{cases} \frac{2mN^2}{N^2-4} & \text{if } |x| \leqslant 1, \\ m|x|^2 + \frac{2m}{(N-2)|x|^{N-2}} + \frac{mN}{N+2}, & \text{if } |x| > 1. \end{cases} \end{aligned}$$

Clearly, $\operatorname{supp}(\rho) = \{x \in \mathbb{R}^N \colon |x| \leqslant 1\}$, and when $|x| \leqslant 1$, we have $\Lambda \equiv \eta$ by (3.20). For $|x| > 1$, note that $\Lambda(x)$ is an increasing function of $|x|$ and equals η when $|x| = 1$; hence, ρ satisfies (3.15), and is a critical point of E.

Note that we can write the repulsive part in (2.5) using the H^{-1}-norm [3], and write the energy as

$$E(\rho) = m\int_{\mathbb{R}^N} |x|^2\rho(x)\,dx + N(N-2)\omega_N\|\rho\|^2_{H^{-1}(\mathbb{R}^N)}.$$

Here, both terms in the energy are strictly convex [4]. Since the energy is strictly convex in every direction and $\rho(x) = \frac{m}{\omega_N}\chi_{B(0,1)}(x)$ is a critical point, it is the global minimizer of the problem (2.1). □

Remark 3.10. When $p = 2 - N$ the repulsive term is always strictly convex as it can be written as the square of the H^{-1}-norm of ρ; however, for $q > 2 - N$, $q \neq 2$, it is difficult to check the convexity of the attractive term due to cross-integral terms in the energy.

Remark 3.11. The scaling of the uniform distribution $(m/\omega_N)\chi_{B(0,1)}$ can be determined by looking at the weak criticality condition (3.23). Indeed, when $q = 2$ and $p = 2 - N$, an explicit calculation shows that for any given $m > 0$ the function

$$\rho_R(x) := \frac{m}{\omega_N R^N}\chi_{B(0,R)}(x)$$

satisfies the weak condition (3.23) if and only if $R = 1$.

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[3] For a function $\rho \in L^2(\mathbb{R}^N)$ the H^{-1}-norm is defined as

$$\|\rho\|^2_{H^{-1}(\mathbb{R}^N)} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho(x)\rho(y)}{N(N-2)\omega_N|x-y|^{N-2}}\,dxdy.$$

[4] A functional $E : \mathcal{A} \to \mathbb{R}$ is strictly convex if for all f and g in $\mathcal{A}$, $f \neq g$, and $t \in (0,1)$ we have $E[tf+(1-t)g] < tE[f] + (1-t)E[g]$.

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