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A Hamiltonian Regularization of the Burgers Equation

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Summary. We consider a quasilinear equation that consists of the inviscid Burgers equation plus $O(\alpha^2)$ nonlinear terms. As we show, these extra terms regularize the Burgers equation in the following sense: for smooth initial data, the $\alpha > 0$ equation has classical solutions globally in time. Furthermore, in the zero-α limit, solutions of the regularized equation converge strongly to weak solutions of the Burgers equation. We present numerical evidence that the zero-α limit satisfies the Oleinik entropy inequality. For all $\alpha \geq 0$, the regularized equation possesses a nonlocal Poisson structure. We prove the Jacobi identity for this generalized Hamiltonian structure.

Key words. Nonlinear evolution equation, Burgers equation, Leray regularization, method of characteristics, singular limit, nonlocal Poisson structure.

MSC numbers. 35L25, 35L65, 35L67, 37K05

1. Introduction

In this paper, we consider the following quasilinear evolution equation:

$$u_t + uu_x - \alpha^2 u_{txx} - \alpha^2 uu_{xxx} = 0, \tag{1}$$

with $\alpha > 0$. By introducing the Helmholtz operator,

$$\mathcal{H} = \mathrm{Id} - \alpha^2 \partial_x^2, \tag{2}$$

we may rewrite (1) as

$$v_t + uv_x = 0 \tag{3}$$

where

$$v = \mathcal{H}u. \tag{4}$$

The main goal of this work is to show that (1) represents a valid regularization of the Burgers equation. That is, we claim that solutions $u^\alpha(x,t)$ of (1) with initial data

$$u^\alpha(x,0) = \mathcal{H}^{-1}v_0(x)$$

converge strongly, as $\alpha \to 0$, to unique entropy solutions of the Cauchy problem for the inviscid Burgers equation

$$u_t + \frac{1}{2}\left(u^2\right)_x \;=\; 0, \tag{5a}$$

$$u(x,0) \;=\; v_0(x). \tag{5b}$$

Before discussing how we establish this claim, let us clarify what is meant by a regularization of a shock-forming hyperbolic equation such as (5a).

Inviscid Burgers and Its Regularizations

We are primarily concerned with the case when $v_0'(x) < 0$ for at least one point $x \in \mathbb{R}$. In this case, regardless of how smooth $v_0(x)$ is, the classical solution $u(x,t)$ of the Burgers equation exists only until a *break time* T (see [Joh90]). Therefore, we work with the weak form of (5). When v_0' has mixed sign, there exists a global weak solution $u(x,t)$ of (5). The solution is not unique unless we impose an additional constraint, which is called an entropy inequality by analogy with gas dynamics. Here we will work with the Oleinik inequality

$$\frac{u(x+a,t)-u(x,t)}{a} < \frac{C}{t}, \tag{6}$$

for every $a > 0$, $t > 0$, and $x \in \mathbb{R}$, where C is a constant that depends only on v_0. Taken together, the system (5)–(6) has a unique weak solution $u(x,t)$ that exists globally in time.

It is well known that if one desires to capture the physically relevant solutions of (5), one can solve the viscous Burgers equation

$$u_t + uu_x \;=\; \nu u_{xx}, \tag{7a}$$

$$u(x,0) \;=\; u_0(x). \tag{7b}$$

By studying the theory for this classical viscous regularization, we may determine the necessary features of an equation that seeks to remedy the breaking of solutions in the inviscid case.

Definition 1. A *regularization* of the inviscid Burgers equation (5a) is an equation of the form

$$u_t + uu_x = \epsilon F(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) \tag{8}$$

satisfying the following criteria:

1. The Cauchy problem for (8) is well posed for all $\epsilon > 0$. That is, for initial data $u(x, 0) = u_0$ in a suitable function space, a unique *classical* solution $u(x, t)$ exists globally in time.
2. As $\epsilon \to 0$, solutions $u^\epsilon(x, t)$ of (8) converge strongly to a weak solution $u(x, t)$ of (5a).
3. This limit $u(x, t)$ is not just any weak solution, but in fact the entropy solution of (5a).

The basic idea is to solve an equation "Burgers + ϵ terms" which has smooth classical solutions that closely approximate discontinuous (weak), physically acceptable solutions of Burgers. It is well known that the viscous regularization, i.e., taking $F = u_{xx}$ in (8), satisfies the definition we have given, even with rough initial data $u_0 \in L^\infty$. Other examples that have appeared in the literature include filtered viscosity, $F = \mathcal{H}^{-1}u_{xx}$, which has been analyzed in [ST92], [LT01], [LM03], as well as hyper-viscosity, $F = (-1)^{n+1}\partial_x^{2n}u$, which has been analyzed in [Tad04]. In both cases, proofs of strong convergence to the entropy solution have been given; in the case of hyper-viscosity, it should be mentioned that an assumption on the L^∞-boundedness of the regularized solution is required.

What is common to all successful regularizations we have seen thus far is that they involve adding a small amount of dissipation, in various guises, to the inviscid equation. The regularization (3) that we analyze substitutes *filtering* for dissipation, as can be seen by rewriting (3) as follows:

$$v_t + \left[\mathcal{H}^{-1}v\right] v_x = 0.$$

We think of the inviscid Burgers equation $v_t + vv_x = 0$ as a transport equation with local transport velocity equal to v itself. Our regularization consists of using a smoothed or filtered version of v—specifically $\mathcal{H}^{-1}v$—in place of v. The smoothing property of $\mathcal{H}^{-1}$ may be seen in physical space, where one finds that $\mathcal{H}^{-1}v$ must have two more weak derivatives than v; the filtering property can be seen in Fourier space, where it is clear that

$$\widehat{\mathcal{H}^{-1}v} = \frac{\widehat{v}}{1 + \alpha^2 k^2}.$$

Historically, it was Leray who, in the context of the incompressible Navier–Stokes equations, first proposed replacing the nonlinear term $(v \cdot \nabla)v$ with a term $(u \cdot \nabla)v$. Here $u = K^\epsilon * v$ for some smoothing kernel K^ϵ. Leray's program consisted in proving existence of solutions for his modified equations and then showing that these solutions converge, as $\epsilon \downarrow 0$, to weak solutions of Navier–Stokes—see [Ler34] for details. The idea of using a Leray-type regularization in lieu of dissipation, for the purposes of capturing shocks in the Burgers equation, was suggested independently by J. E. Marsden, K. Mohseni, and E. S. Titi, and perhaps others of whom we are not aware. As described in [MZM06], initial numerical simulations and physical arguments of K. Mohseni and H. Zhao led them to suggest a family of models, including (1), as a proper regularization of the inviscid Burgers equation.

There is another way to view the regularization (1). Using the definition (2) of $\mathcal{H}$, we may verify that

$$u_t + uu_x = -\frac{3}{2}\alpha^2\mathcal{H}^{-1}\left(u_x^2\right)_x \tag{9}$$

is formally equivalent to (1). The right-hand side of (9) represents a *nonlinear* smoothing term that differs from standard viscosities, which are all linear in u. Currently, we are unaware of previous works which have proved that either the filtering approach or the right-hand side term in (9) may be used to regularize the inviscid Burgers equation, in the sense of Definition 1. We are also unaware of regularizations that are Hamiltonian in any sense, unlike (1), which is Hamilton's equation for a certain nonlocal Poisson structure.

Of course, there are numerous Hamiltonian PDE that can be written in the form (8) and "regularize" the Burgers equation in a certain limited sense: the equations

$$u_t + uu_x + \epsilon u_{xxx} = 0 \qquad \text{Korteweg–de Vries (KdV)} \tag{10}$$

$$u_t + uu_x - \epsilon u_{xxt} = 0 \qquad \text{Benjamin–Bona–Mahony (BBM)} \tag{11}$$

possess smooth classical solutions and hence satisfy criterion 1 of Definition 1. However, as $\epsilon \to 0$, solutions of these Hamiltonian PDE *fail* to converge, even weakly, to weak solutions of the inviscid Burgers equation. The equation

$$u_t + uu_x + \epsilon uu_{xxx} = 0, \tag{12}$$

the continuum limit of a semidiscrete scheme considered in [GL88], also fails to produce solutions that converge as $\epsilon \to 0$, even weakly, to weak Burgers solutions. Examining (11)–(12), one notices that *individually* taking the $O(\alpha^2)$ terms from (1) fails to yield a satisfactory regularization of the Burgers equation.

Summary of Results

To support the claim made earlier, we will prove that the regularization (1) satisfies criteria 1 and 2 of Definition 1. We will provide numerical evidence that criterion 3 of Definition 1 is satisfied.

More specifically, in the present work, we prove well-posedness of the Cauchy problem for (3), establishing a classical solution $v^\alpha(x,t)$ to (3) that exists globally in time, given initial data $v_0 \in W^{2,1}(\mathbb{R})$. We prove that as $\alpha \to 0$, the sequence $u^\alpha = \mathcal{H}^{-1}v^\alpha$ converges strongly to a function u that is a global-in-time weak solution of the inviscid Burgers equation. We provide numerical evidence that the resulting limit function u satisfies a form of the Oleinik entropy inequality. Finally, we prove that (1) is Hamiltonian, with respect to a particular Poisson bracket.

Previous Results on Equation (1)

Equation (1) has appeared previously in the literature, as the $b = 0$ member of the b-family described in [DHH03]:

$$v_t + uv_x + bu_xv = 0. \tag{13}$$

Various results regarding the complete integrability (for $b = 2$ and $b = 3$) and traveling wave solutions of (13) may be found in [DHH03], [HW03], [HH05], [HS03], [DGH03], [DGH04], [MN02], [CHT04]. In what follows, we discuss the results from this collection that specifically deal with the $b = 0$ case of (13).

Physical motivation for the b-family is provided in [DGH03, DGH04], which show that (13) is an asymptotically equivalent approximation of the shallow water equations. In [DHH03], the b-family is realized as the Euler–Lagrange equation corresponding to a certain Lagrangian density. As the authors point out, this Lagrangian structure breaks down when $b = 0$. The authors do propose a Hamiltonian structure that appears well defined for the $b = 0$ case, though they do not prove here that the proposed structure in fact satisfies the requirements for Hamiltonian operators as described in, e.g., [Olv93].

The Hamiltonian structure of the b-family (13) is given by (see [HW03]):

$$v_t = -b^2 \mathcal{B} \frac{\delta H}{\delta v}, \qquad H = \frac{1}{b-1} \int v \, dx, \tag{14a}$$

$$\mathcal{B} = v^{1-1/b} \partial_x v^{1/b} (\partial_x - \alpha^2 \partial_x^3)^{-1} v^{1/b} \partial_x v^{1-1/b}. \tag{14b}$$

As the authors of [HW03] state, "when $b = 1$ the Hamiltonian must be modified; for $b = 0$ the operator $\mathcal{B}$ can be redefined." In [HW03], they prove that except in these special cases, the functional/operator pair given in (14) satisfies the Jacobi identity and is a valid Hamiltonian structure for the b-family (13). In this paper, we show that the proper redefinition of (14) in the $b = 0$ case is also a valid Hamiltonian structure. Hence (1) is Hamiltonian in a certain sense.

For another clue about why the $b = 0$ case of (13) is special, let us look at dispersion relations in the $b = 0$ versus $b > 0$ cases. First we consider sinusoidal perturbations about a constant solution $u = u_0$ and write the dispersion relation for (1):

$$\omega(k) = u_0 k. \tag{15}$$

Hence (1) is not a dispersive wave equation. Meanwhile, the dispersion relation for the whole b-family (13) is

$$\omega(k) = u_0 k + u_0 \frac{bk}{1 + \alpha^2 k^2}. \tag{16}$$

The b-family is dispersive for $b > 0$; furthermore, for the $b > 0$ members of the family (13), the zero-α limit is a zero-dispersion limit. We expect that high-frequency oscillations typical of the zero-dispersion limit of KdV will play a role in the zero-α limits of the b-family for all $b > 0$. Again, the $b = 0$ equation under consideration in this paper is not dispersive, and we prove that it does not produce oscillations in the zero-α limit.

Outline of this Paper

We study the initial-value problem of (1) in Section 2 and prove well-posedness of solutions for certain initial data. Next, we consider the $\alpha \to 0$ limit. In Section 3 we show that solutions of (1) converge strongly in the zero-α limit to weak solutions of the inviscid Burgers equation. In Section 4, we provide numerical evidence that the weak solution that is selected in the $\alpha \to 0$ limit is indeed the physically relevant, entropy solution. We discuss the Hamiltonian structure of (1) in Section 5. Finally, in Section 6, we discuss future directions in which we plan to take this line of research.

2. Initial-Value Problem

In this section we study the existence, uniqueness and regularity of solutions of the Cauchy problem for (1) (or equivalently, (3), (4)) for certain classes of initial data. Making use of a particular dilation symmetry, we may restrict attention to the $\alpha = 1$ system:

$$v_t + uv_x = 0, \tag{17a}$$

$$u - u_{xx} = v, \tag{17b}$$

$$v(x,0) = v_0(x). \tag{17c}$$

Given a solution $u(x,t)$ of (17a)-(17b), we may construct

$$\tilde{u}(x,t) = u\left(\frac{x}{\alpha}, \frac{t}{\alpha}\right). \tag{18}$$

Then it is easy to check that $\tilde{u}$ solves (1) for any $\alpha > 0$.

Material Version of the Regularized Equation

To prove the existence and uniqueness of solutions of (17) we shift the view from the spatial to the material picture. Suppose that (17) holds for a smooth function $v(x,t)$. Then we may solve (17b) for u and define the associated material map η as the solution of

$$\partial_t \eta(X,t) = u(\eta(X,t),t), \tag{19}$$

subject to $\eta(X,0) = X$. Here, X denotes the Lagrangian coordinate (particle label). It is clear that (17) becomes, simply,

$$\frac{d}{dt}\left[v(\eta(X,t),t)\right] = 0,$$

which implies

$$v(\eta(X,t),t) = v(\eta(X,0),0) = v_0(X). \tag{20}$$

Remark. The curves $\eta(X,t)$ are commonly called "characteristics." The condition

$$\partial_X \eta(X,t) \neq 0, \tag{21}$$

is simply the statement that characteristics do not cross. Provided (21) holds for all X and t, the map $X \mapsto \eta(X,t)$ is a diffeomorphism for each fixed t. Hence, the map η can be inverted and v, defined by

$$v(x,t) = v_0(\eta^{-1}(x,t)),$$

is a global smooth solution of (17), provided the initial data v_0 is smooth.

Differentiate (20) with respect to X,

$$\partial_x v(\eta(X,t),t)\partial_X \eta(X,t) = v_0'(X), \tag{22}$$

to observe that the condition (21) is equivalent to the non-blow-up in finite time of $\|v_x\|_{L^\infty}$. Below we will prove that $\|v_{xx}\|_{L^1}$ cannot blow-up in finite time, which implies the non-blow-up of $\|v_x\|_{L^\infty}$.

A Priori Estimates

Due to (20), it is clear that, as long as the problem (17) has a smooth solution, it satisfies

$$\|v\|_{L^\infty} = \|v_0\|_{L^\infty}. \tag{23}$$

The Green's function of the Helmholtz operator (2) with $\alpha = 1$ is

$$G(x) = \frac{1}{2}\exp(-|x|).$$

Note that

$$\|G\|_{L^1} = 1 \text{ and } \|G_x\|_{L^1} = 1.$$

We invert (17b) by convolving

$$u(x,t) = G * v := \frac{1}{2}\int_{\mathbb{R}} \exp(-|x-y|)v(y,t)\,dy. \tag{24}$$

Young's inequality guarantees that $v \in L^p$ implies $u \in L^p$. In particular, for $v \in L^\infty$, we may use (24) to conclude $u \in L^\infty$, and because $G_x \in L^1$, we know that u_x exists and can be computed via

$$u_x = G_x * v.$$

Using these facts, the following estimates become readily available:

$$\|u_x\|_{L^\infty} \le \|G_x\|_{L^1}\|v\|_{L^\infty} \quad = \|v_0\|_{L^\infty} \tag{25}$$

$$\|u\|_{L^\infty} \le \|G\|_{L^1}\|v\|_{L^\infty} \quad = \|v_0\|_{L^\infty} \tag{26}$$

$$\|u_{xx}\|_{L^\infty} = \|u - v\|_{L^\infty} \ \le 2\|v_0\|_{L^\infty}. \tag{27}$$

L^1 estimate on v_x. Differentiate (17a) with respect to x, multiply by $\mathrm{sgn}(v_x)$ and integrate over the x domain to obtain:

$$\int |v_x|_t\,dx + \int (uv_x)_x\,\mathrm{sgn}(v_x)dx = 0.$$

The second term in the left-hand side of the equation above is zero. Hence,

$$\|v_x(\cdot,t)\|_{L^1} = \|v_0'\|_{L^1}, \quad \text{for all } t > 0. \tag{28}$$

L^1 **estimate on** v_{xx}**.** Differentiate (17a) twice with respect to x, multiply by $\mathrm{sgn}(v_{xx})$ and integrate over the x domain to obtain:

$$\begin{aligned}\int |v_{xx}|_t \, dx &= -\int (uv_x)_{xx} \,\mathrm{sgn}(v_{xx}) dx \\ &= -\int (uv_{xx})_x \,\mathrm{sgn}(v_{xx}) dx - \int u_{xx} v_x \,\mathrm{sgn}(v_{xx}) dx \\ &\quad - \int u_x v_{xx} \,\mathrm{sgn}(v_{xx}) dx. \end{aligned} \tag{29}$$

The first term in the right-hand side of (29) is zero. We estimate the remaining two terms as follows:

$$-\int u_{xx} v_x \,\mathrm{sgn}(v_{xx}) dx \leq \int |u_{xx}||v_x| dx \leq \|u_{xx}\|_{L^\infty} \|v_x\|_{L^1},$$

and

$$-\int u_x v_{xx} \,\mathrm{sgn}(v_{xx}) dx \leq \int |u_x||v_{xx}| dx \leq \|u_x\|_{L^\infty} \|v_{xx}\|_{L^1}.$$

We know that $\|u_x\|_{L^\infty}$, $\|u_{xx}\|_{L^\infty}$ and $\|v_x\|_{L^1}$ are bounded by constants that depend on the initial data only (see (25), (27) and (28)). Hence, by using the last two estimates in (29), we have

$$\frac{d}{dt}\|v_{xx}\|_{L^1} \leq C_1 \|v_{xx}\|_{L^1} + C_2,$$

where C_1 and C_2 depend only on the initial data. Now, Gronwall's lemma gives the L^1-boundedness of v_{xx}, for finite times. The Sobolev imbedding (in the notation of [Ada75, Thm. 5.4])

$$W^{1,1}(\mathbb{R}) \to L^\infty(\mathbb{R})$$

then guarantees the L^∞-boundedness of v_x, for finite times. As mentioned earlier in (22), this non-blow-up condition on $\|v_x\|_{L^\infty}$ is precisely what is needed to show that characteristics cannot cross in finite time. Using standard arguments, this implies that characteristics can be extended uniquely from the initial data for any time interval $[0, T]$.

For the estimates of this section to make sense, we require that v_0 possess the regularity $v_0 \in C^1 \cap L^\infty$, and also that v_0 have two weak derivatives that satisfy $v_0', v_0'' \in L^1$. By virtue of the Sobolev imbedding[1]

$$W^{2,1}(\mathbb{R}) \to C_B^1(\mathbb{R}),$$

it is sufficient to take $v_0 \in W^{2,1}$. Hence we have proved:

Theorem 1. *Given initial data* $v_0 \in W^{2,1}(\mathbb{R})$*, there exists a unique global solution* $v(x, t) \in W^{2,1}(\mathbb{R})$ *to the initial-value problem (17).*

[1] In what follows, $C_B^1(\mathbb{R}) = \{f \in C^1(\mathbb{R}) \text{ such that } \sup_{x\in\mathbb{R}} |f(x)| + |f'(x)| < \infty\}$.

3. The $\alpha \to 0$ Limit

Let us now examine in a different context the Cauchy problem

$$v_t + uv_x = 0, \tag{30a}$$

$$u - \alpha^2 u_{xx} = v, \tag{30b}$$

$$v(x,0) = v_0(x), \tag{30c}$$

subject to initial data $v_0 \in W^{2,1}(\mathbb{R})$. Let $v^\alpha(x,t)$ denote the unique solution to the Cauchy problem, which exists based on the results of Section 2. Now we can formulate the question: what happens to $u^\alpha(x,t) = \mathcal{H}^{-1}v^\alpha(x,t)$ in the limit as $\alpha \to 0$? We may think of this limiting process as repeatedly solving the Cauchy problem with fixed initial data v_0 while taking values of α from a sequence $\{\alpha_n\}$, where $\alpha_n \downarrow 0$ as $n \to \infty$.

Initial Data

It is important to remember that as we repeatedly solve (30) with decreasing values of α, the initial data v_0 stays fixed. How does this affect $u^\alpha(x,0)$? To answer this, simply note that using the Green's function

$$G^\alpha(x) := \frac{1}{2\alpha}\exp(-|x|/\alpha), \tag{31}$$

we may write

$$u_0^\alpha(x) = (G^\alpha * v_0)(x).$$

Note that $\|G^\alpha\|_{L^1} = 1$, while v_0 is bounded and continuous, for all $\alpha > 0$. Then it is a standard property of convolutions (see [Fol99, Thm. 8.14]) that as $\alpha \to 0$,

$$u_0^\alpha \to v_0$$

uniformly on compact subsets of $\mathbb{R}$.

Conservation Properties/Estimates

Recall from the previous section that η gives particle trajectories corresponding to the velocity field u. We established in (20) that v is constant along η.

Our next step is use the a priori estimates from the previous section to prove estimates uniform in α for solutions $u^\alpha(x,t)$ to (30) with $v_0 \in W^{2,1}(\mathbb{R})$. The uniform L^∞ and BV bounds that we are about to show will enable us to pass to the $\alpha \to 0$ limit by a standard conservation law compactness argument; we then establish that the resulting limit is a weak solution of the inviscid Burgers equation.

Proposition 1. *Given initial data $v_0 \in W^{2,1}(\mathbb{R})$ for the Cauchy problem (30), the resulting solution $v^\alpha(x,t)$ satisfies*

$$\|v^\alpha(\cdot,\cdot)\|_{L^\infty} = \|v_0(\cdot)\|_{L^\infty}, \tag{P1}$$

$$\|v_x^\alpha(\cdot,t)\|_{L^1} = \|v_0'(\cdot)\|_{L^1}, \tag{P2}$$

$$\text{T.V.}\, v^\alpha(\cdot,t) = \text{T.V.}\, v_0. \tag{P3}$$

Proof. The relations (P1) and (P2) for $\alpha = 1$ were established in Section 2 (see (23) and (28)). Clearly, the same proof applies for a general α. As regards (P3), this follows immediately from (P2) together with the fact that for a smooth function f,

$$\text{T.V.}\, f \;=\; \int_{\mathbb{R}} \left| f'(x) \right| \, dx. \qquad \square$$

Proposition 2. *Given initial data $v_0 \in W^{2,1}(\mathbb{R})$ for the Cauchy problem (30), the solution $v^\alpha(x,t)$ may be used to define the function $u^\alpha(x,t) = \mathcal{H}^{-1} v^\alpha(x,t)$. Then $u^\alpha(x,t)$ satisfies*

$$\|u^\alpha(\cdot,\cdot)\|_{L_\infty} \;\le\; M_1, \tag{H1}$$

$$\|u^\alpha(x+h,t) - u^\alpha(x,t)\|_{L^1} \;\le\; M_2|h| \quad \textit{for any } h \in \mathbb{R}, \tag{H2}$$

$$\|u^\alpha(\cdot,t+k) - u^\alpha(\cdot,t)\|_{L_1} \;\le\; M_3 k, \quad \textit{for any } k > 0, \tag{H3}$$

for $t \in [0,T]$. Here, M_1 is independent of α, M_2 is independent of t, h, and α, and M_3 is independent of t, k and α.

Proof. In what follows we recall the Green's function G^α defined in (31) and use the fact that for all x,

$$\|G^\alpha(x-y)\|_{L^1} = \frac{1}{2\alpha}\int_{\mathbb{R}} e^{-|x-y|/\alpha}\, dy = 1.$$

Then starting with $u^\alpha = G^\alpha * v^\alpha$, we use (P1) to estimate

$$\begin{aligned} |u^\alpha(x,t)| \;&\le\; \frac{1}{2\alpha}\int_{\mathbb{R}} e^{-|x-y|/\alpha} |v^\alpha(y,t)|\, dy \\ &\le\; \|v_0\|_{L^\infty} \|G^\alpha(x-y)\|_{L^1} = \|v_0\|_{L_\infty}, \end{aligned}$$

proving (H1).

To prove (H2), we estimate

$$\begin{aligned} \int_{\mathbb{R}} |u^\alpha(x+h,t) - u^\alpha(x,t)|\, dx \;&\le\; \frac{1}{2\alpha}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-|x-y|/\alpha} |v^\alpha(y+h,t) - v^\alpha(y,t)|\, dy\, dx \\ &=\; \int_{\mathbb{R}} |v^\alpha(y+h,t) - v^\alpha(y,t)|\, dy\, \|G^\alpha(x-y)\|_{L^1} \\ &=\; \int_{\mathbb{R}} |v^\alpha(x+h,t) - v^\alpha(x,t)|\, dx. \end{aligned} \tag{32}$$

Now, (H2) follows from (32), the inequality

$$\int_{\mathbb{R}} |v^\alpha(x+h,t) - v^\alpha(x,t)| \le |h|\, \text{T.V.}\, v^\alpha(\cdot,t),$$

and (P3).

Finally, to prove (H3) we start from the following estimate, derived in the same way as (32):

$$\int_{\mathbb{R}} |u^\alpha(x, t+k) - u^\alpha(x, t)|\, dx \le \int_{-\infty}^{\infty} |v^\alpha(x, t+k) - v^\alpha(x, t)|\, dx. \tag{33}$$

By integrating (3) from t to $t + k$ ($k > 0$), we have

$$\begin{aligned}\int_{\mathbb{R}} |v^\alpha(x, t+k) - v^\alpha(x, t)|\, dx \;&\le\; \int_{\mathbb{R}} \int_t^{t+k} |u^\alpha(x, s) v_x^\alpha(x, s)|\, ds\, dx \\ &\le\; \|u^\alpha\|_{L^\infty} \int_t^{t+k} \|v_x^\alpha(\cdot, s)\|_{L^1}\, ds \\ &\le\; M_1 \|v_0'\|_{L^1} k,\end{aligned} \tag{34}$$

where we used (P2) and (H1) in the last inequality. Combining (33) with (34), we have the desired result. □

Strong Convergence to a Weak Solution of Burgers

Using the estimates given above, we may prove the following:

Theorem 2. *Suppose we solve the Cauchy problem (30) with initial data $v_0 \in W^{2,1}(\mathbb{R})$. Using the solution v^α, let us define $u^\alpha = \mathcal{H}^{-1} v^\alpha$ in the usual way. Then, as $\alpha \to 0$, passing if necessary to a subsequence, there exists a function $u(x, t)$ such that*

$$u^\alpha \to u \text{ in } C([0, \infty); L^1_{\text{loc}}(\mathbb{R})).$$

The function u is a global weak solution of the initial-value problem (5) for the inviscid Burgers equation.

Proof. The first part of the theorem concerns compactness, i.e., strong convergence of u^α in the zero-α limit. The three uniform estimates proved in Proposition 2 are precisely the conditions of the L^1 compactness theory for conservation laws. (See [HR02, Thm. A.8] or [Smo83, Thm. 19.9] for modern accounts of this.) The specific result is that there exists a subsequence $\alpha_j \to 0$ such that $\{u^{\alpha_j}(t)\}$ converges strongly to a function $u(x, t)$, where $u(\cdot, t) \in L^1_{\text{loc}}(\mathbb{R})$ for each $t \ge 0$. The convergence is in $C([0, \infty); L^1_{\text{loc}}(\mathbb{R}))$.

For the second half of the theorem, we go back to equation (1), which we repeat here:

$$u_t^\alpha + u^\alpha u_x^\alpha - \alpha^2 u_{txx}^\alpha - \alpha^2 u^\alpha u_{xxx}^\alpha = 0. \tag{35}$$

We wish to prove that the α^2 terms

$$\alpha^2 u_{txx}^\alpha + \alpha^2 u^\alpha u_{xxx}^\alpha,$$

converge weakly to 0 as $\alpha \to 0$. Suppose we have shown this; then, we may multiply (35) by a test function φ that is compactly supported in $\mathbb{R} \times [0, \infty)$ and integrate in space and time. Now taking $\alpha \to 0$, we will find that the order α^2 terms vanish, and we are left with a function u that satisfies

$$\int_0^\infty \int_{\mathbb{R}} u\varphi_t + \frac{1}{2}u^2\varphi_x \, dx \, dt = 0,$$

for all compactly supported φ. This is precisely the statement that u is a global weak solution of the inviscid Burgers equation, and would prove the theorem.

For the first α^2 term from (35), we have, for any compactly supported φ,

$$\alpha^2 \int_0^T \int_{\mathbb{R}} u^\alpha_{txx}\varphi \, dx \, dt = -\alpha^2 \int_0^T \int_{\mathbb{R}} u^\alpha \varphi_{txx} \, dx \, dt.$$

Using the convergence of the sequence u^α, it is clear that this term converges to 0 as $\alpha \to 0$. For the second α^2 term from (35), we may derive using integration by parts

$$\alpha^2 \int_0^T \int_{\mathbb{R}} u^\alpha u^\alpha_{xxx}\varphi \, dx \, dt = \frac{1}{4}\alpha^2 \int_0^T \int_{\mathbb{R}} (u^\alpha)^2 \varphi_{xxx} \, dx \, dt - \frac{3}{2}\alpha^2 \int_0^T \int_{\mathbb{R}} u^\alpha u^\alpha_{xx}\varphi_x \, dx \, dt. \tag{36}$$

By using the boundedness and the convergence of u^α, we conclude that the first term on the right-hand side of (36) vanishes in the $\alpha \to 0$ limit. Regarding the second term, by considering the boundedness of u^α, it is enough to show that

$$\alpha^2 \int_0^T \int_K |u^\alpha_{xx}| dx \to 0,$$

for any compact K. We have

$$\begin{aligned}
\int_0^T \int_K |\alpha^2 u^\alpha_{xx}| \, dx &= \int_0^T \int_K |u^\alpha - v^\alpha| \, dx \\
&= \int_0^T \int_K \left| \frac{1}{2\alpha} \int_{\mathbb{R}} e^{-|x-y|/\alpha} v^\alpha(y,t) \, dy - v^\alpha(x,t) \right| \, dx \, dt.
\end{aligned}$$

Here, we used $u^\alpha = G^\alpha * v^\alpha$ to obtain the second equality. Integrating by parts, we get

$$\frac{1}{2\alpha} \int_{\mathbb{R}} e^{-|x-y|/\alpha} v^\alpha(y,t) \, dy = v^\alpha(x,t) + \frac{1}{2} \int_{\mathbb{R}} \operatorname{sgn}(y-x) e^{-|y-x|/\alpha} v^\alpha_y(y,t) \, dy.$$

Continuing, we find

$$\begin{aligned}
\int_0^T \int_K |\alpha^2 u^\alpha_{xx}| \, dx \, dt &\le \frac{1}{2} \int_0^T \int_K \int_{\mathbb{R}} e^{-|y-x|/\alpha} |v^\alpha_y(y,t)| \, dy \, dx \, dt \\
&= \frac{1}{2} \int_0^T \int_K |v^\alpha_y(y,t)| \, dy \, dt \int_{\mathbb{R}} e^{-|y-x|/\alpha} \, dx \\
&= \alpha \int_0^T \int_K |v^\alpha_y(y,t)| \, dy \, dt.
\end{aligned}$$

From (P2), we conclude that the term

$$\alpha \int_0^T \int_K |v^\alpha_y(y,t)| \, dy \, dt$$

is of order $O(\alpha)$ and hence, goes to 0 as $\alpha \to 0$. This completes the argument. □

4. Entropy/Numerics

Using a standard finite-difference scheme, we solve the initial-value problem (30) numerically with an eye toward checking the Oleinik entropy inequality (6). Here we simply describe the numerical scheme, deferring discussion of its convergence properties to future work. Then we discuss various numerical results for both short- and long-time simulations.

Numerical Scheme

Beginning with system (30), we truncate the spatial domain to $[-a, a]$. Because the domain is now finite, we must impose artificial boundary conditions; we impose the condition that v vanishes for $|x| > a$. We discretize the domain $[-a, a]$ using an equispaced grid with N grid points. Let us denote this grid by $x_i = -a + (i-1)\Delta x$, where $i = 1, \ldots, N$, and the grid spacing is given by $\Delta x = 2a/(N-1)$.

On this discrete domain, we consider the evolution in time of the vector $\mathbf{v}(t) = (v_1(t), \ldots, v_N(t))$. We will suppose that $v_i(t) \approx v(x_i, t)$.

Following [Ise96], we define two basic operators on $\mathbb{R}^N$:

$$\mathbf{z} \mapsto \Delta_0^2 \mathbf{z}, \quad \left(\Delta_0^2 \mathbf{z}\right)_k = z_{k+1} - 2z_k + z_{k-1}, \tag{37}$$

$$\mathbf{z} \mapsto \Gamma_0 \mathbf{z}, \quad (\Gamma_0 \mathbf{z})_k = \frac{1}{2}\left(z_{k+1} - z_{k-1}\right). \tag{38}$$

Here we use the convention that $z_k = 0$ for $k < 1$ and for $k > N$. This corresponds to the artificial boundary conditions discussed above. Note that the operators (37)–(38) are in fact linear transformations of $\mathbb{R}^N$ and may be written in matrix form. Now in terms of the operators (37)–(38), we may write the standard finite-difference approximations to the first- and second-derivative operators ∂_x and ∂_x^2:

$$D^1 = \frac{1}{\Delta x}\Gamma_0\left[\mathrm{Id} - \frac{1}{6}\left(\Delta_0^2\right) + \frac{1}{30}\left(\Delta_0^2\right)^2\right] + \mathcal{O}(\Delta x^5), \tag{39}$$

$$D^2 = \frac{1}{\Delta x^2}\left[\left(\Delta_0^2\right) - \frac{1}{12}\left(\Delta_0^2\right)^2 + \frac{1}{90}\left(\Delta_0^2\right)^3\right] + \mathcal{O}(\Delta x^6). \tag{40}$$

With this notation, it is clear that the semidiscrete form of (30) is

$$\mathbf{v}_t = -\left[\left(\mathrm{Id} - \alpha^2 D^2\right)^{-1}\mathbf{v}\right] D^1 \mathbf{v}, \tag{41a}$$

$$v_j(0) = v(x_j, 0), \tag{41b}$$

where $v(x, 0)$ is the initial data for the continuum problem and where concatenation of vectors means component-wise multiplication, i.e., $(\mathbf{ab})_k = a_k b_k$. The first-order ODE (41) can now be solved numerically using the time-stepping algorithm of one's choice—we used a high-order explicit Runge–Kutta method.

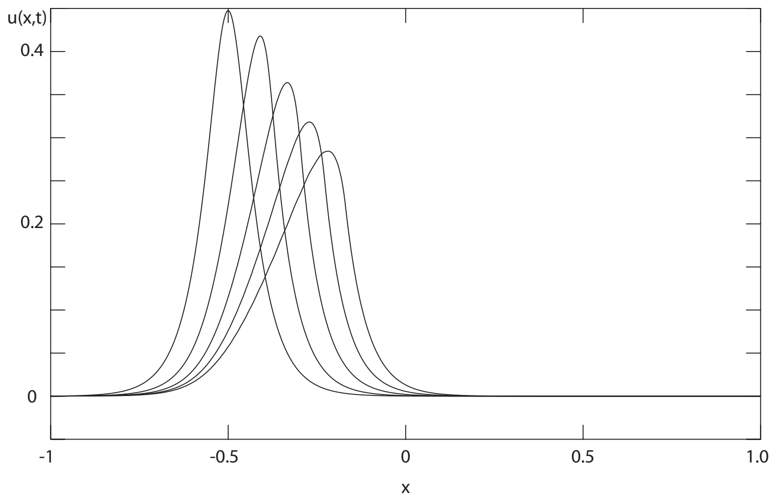

Fig. 1. The numerical solution $u^\alpha(x, t)$ of (30) for $\alpha = 0.3$ with initial data (42). The tallest curve, with a peak at $x = -0.5$, is the solution at $t = 0$. From left to right, we then have the solutions at $t = 1.25$, $t = 2.5$, $t = 3.75$, and $t = 5$. As time passes, the height of the pulse decays while its width increases.

Norm Decay of Solutions

First we present results for the following choice of initial data

$$v(x, 0) = \operatorname{sech}^2\left(\frac{x + 1/2}{1/5}\right), \tag{42}$$

for $\alpha = 0.3$. With this choice of initial data, $v(x, 0) \in W^{2,1}$, so we are within the bounds of our well-posedness and convergence theory. We solve the problem using $N = 1024$ grid points. See Figure 1 for snapshots of the solution $u(x, t)$ at $t = 0$, $t = 1.25$, $t = 2.5$, $t = 3.75$, and $t = 5$. The initial profile does not shock or develop any singularities. Instead, it decays steadily in a rather similar fashion as the solution of the viscous Burgers equation (7a). To see this decay in three norms, we use the numerically computed solution to compute $\|u(\cdot, t)\|_{L^1}$, $\|u(\cdot, t)\|_{L^2}$, and $\|u(\cdot, t)\|_{L^\infty}$ as functions of time t. The results are plotted in Figure 2, clearly showing the decay. Here we see that the L^1 norm of u stays constant in time, i.e.,

$$\|u(\cdot, t)\|_{L^1} = \|u_0\|_{L^1}. \tag{43}$$

This is a simple consequence of the fact that we chose $v_0 > 0$. For, when $v_0 > 0$, we know that $v(x, t) > 0$ for all $x \in \mathbb{R}$, $t > 0$. Then, using $u = G^\alpha * v$, we may deduce that $u(x, t) > 0$ for all $x \in \mathbb{R}$, $t > 0$ as well, so

$$\int_{\mathbb{R}} u(x, t)\, dx = \|u(\cdot, t)\|_{L^1}$$

for all t. The fact that $\int u$ is conserved is a simple consequence of the conservation law form (9), which then explains why we observed (43). Note also from Figure 2 that both

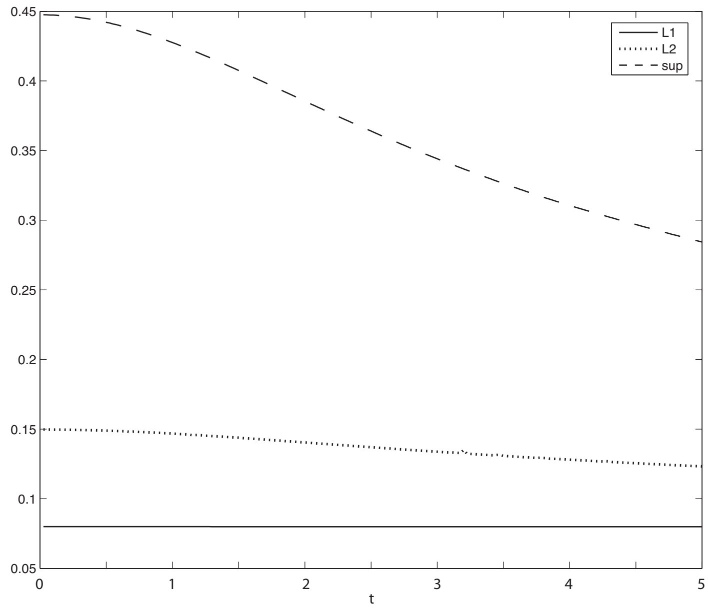

Fig. 2. $\|u(\cdot,t)\|_{L^1}$, $\|u(\cdot,t)\|_{L^2}$, and $\|u(\cdot,t)\|_{L^\infty}$ as functions of t for the solution to (30) with initial data (42) and $\alpha = 0.3$.

the L^2 and L^∞ norms of u are strictly decreasing in time. We showed the L^∞ decay property in the previous section—see Proposition 2. However, at this time, we have no analytical method for deriving a uniform L^2 decay law such as what is seen in Figure 2.

Entropy Inequality: Numerical Evidence

In the previous section, we established a basic convergence theory for (1). That is, we choose initial data $v_0 \in W^{2,1}$, solve (1), and label the solution as u^α. Then we know that a subsequence of u^α converges, in the zero-α limit, to a function u. We know one more thing: this function u is a weak solution of the inviscid Burgers equation (5) with initial data v_0.

At the time of writing, this is where rigorous analysis ends. This is unfortunate, in light of the fact that there are many weak solutions of the inviscid Burgers equation (5) with initial data v_0—the unique, physically relevant solution, is the one that satisfies the Oleinik inequality (6).

We will investigate numerically the validity of

$$\sup_x u_x^\alpha(x,t) < \frac{C}{t}, \tag{44}$$

where C does not depend on α. Fortunately, there is plenty of numerical evidence that (44) holds uniformly in α. By implication, this is evidence that the strong limit u of solutions of (1) does in fact satisfy the Oleinik inequality (6). Let us discuss some of this numerical evidence.

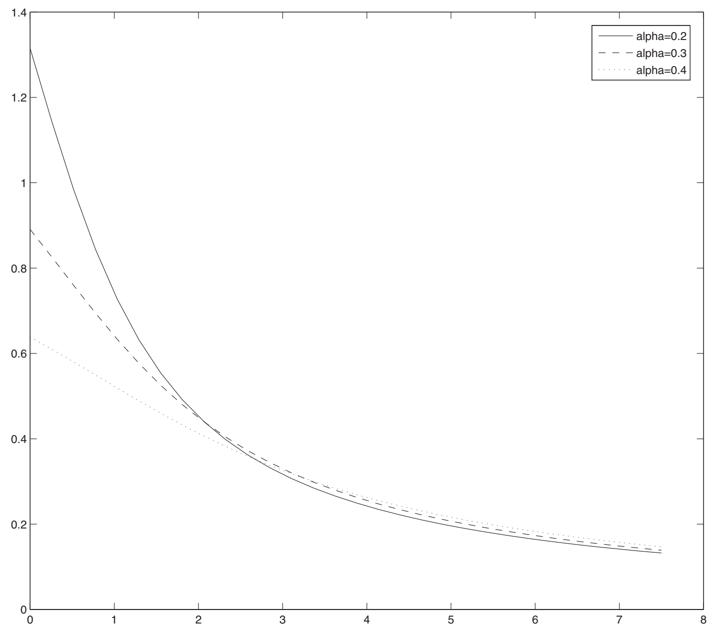

Fig. 3. Plot of $\sup_x u_x^\alpha(x, t)$ as a function of t for three decreasing values of α, from $t = 0$ to $t = 7$.

We repeatedly solve (30) with the initial data (42) using successively smaller α values: $\alpha = 0.4$, $\alpha = 0.3$, and $\alpha = 0.2$. We then plot the quantity

$$m^\alpha(t) := \sup_x u_x^\alpha(x, t) \tag{45}$$

as a function of t for each of the three values of α. First we present Figure 3 which shows (45) from $t = 0$ until $t = 7$. The same quantity (45) from $t = 5$ until $t = 50$ is plotted in Figure 4. Both plots lead one to believe that as $\alpha \to 0$, the curves $m^\alpha(t)$ are uniformly bounded by a curve of the form C/t. The evidence becomes clearer when we consider the same data on logarithmic axes. Taking the logarithm of both sides of (44), we obtain for $t > 1$,

$$\frac{\log\left(\sup_x u_x^\alpha(x, t)\right)}{\log t} < \frac{\log C}{\log t} - 1, \tag{46}$$

where C must not depend on α. With this in mind, we examine Figure 5, which shows the same data as Figure 4 now plotted on a log-log scale. The numerically computed slope of the linear part of this plot is less than -1.25, meaning that the numerically computed solutions $u^\alpha(x, t)$ all satisfy

$$\frac{\log\left(\sup_x u_x^\alpha(x, t)\right)}{\log t} < -1 < -1 + \frac{\log C}{\log t}, \tag{47}$$

for any $C > 1$. Let us remark that we have run the same numerical test with different choices of initial data, resolution, and values of α. In all cases, we find that the numerically computed solutions satisfy (47).

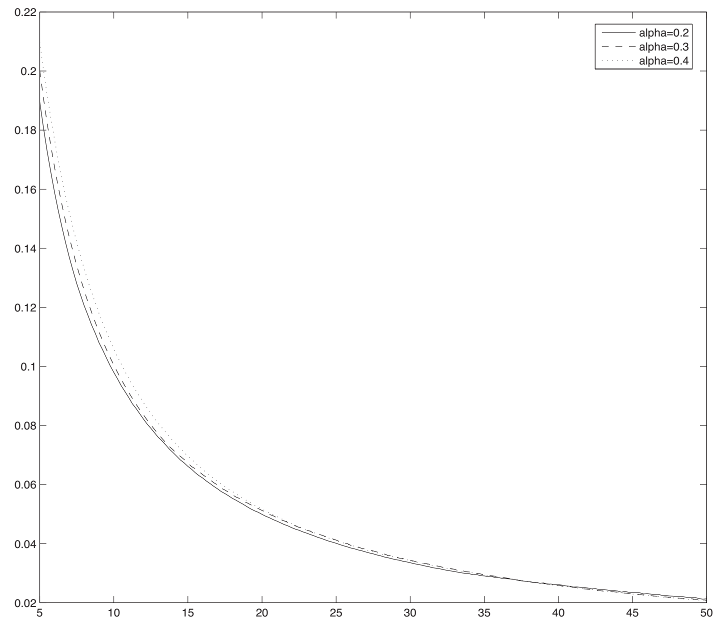

Fig. 4. Plot of $\sup_x u_x^\alpha(x, t)$ as a function of t for three decreasing values of α, from $t = 5$ until $t = 50$.

There is solid numerical evidence that the solutions $u^\alpha(x, t)$ satisfy (44). Because we have not found any evidence that falsifies this claim, we theorize that the limit function $u(x, t)$ is indeed an entropy solution of the inviscid Burgers equation.

5. Geometric Structure

Consider the functional $H\colon L^1(\mathbb{R}) \to \mathbb{R}$ defined by

$$H(v) = \int_{\mathbb{R}} v\, dx, \tag{48}$$

and the operator

$$\mathfrak{D} = -v_x \left(\partial_x - \alpha^2 \partial_x^3\right)^{-1} v_x. \tag{49}$$

This functional/operator pair is the $b = 0$ case of the Hamiltonian structure given in [DHH03] for the b-family (see (13)). Using these two objects, we write the infinite-dimensional generalization of Hamilton's equation:

$$v_t = \mathfrak{D} \frac{\delta H}{\delta v}. \tag{50}$$

Here $\delta H/\delta v$ is the *functional derivative*, defined by

$$\left\langle \frac{\delta H}{\delta v}, \delta v \right\rangle (v) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} H\left(v + \epsilon \delta v\right),$$

where $\langle \cdot\,, \cdot \rangle$ is the natural pairing.

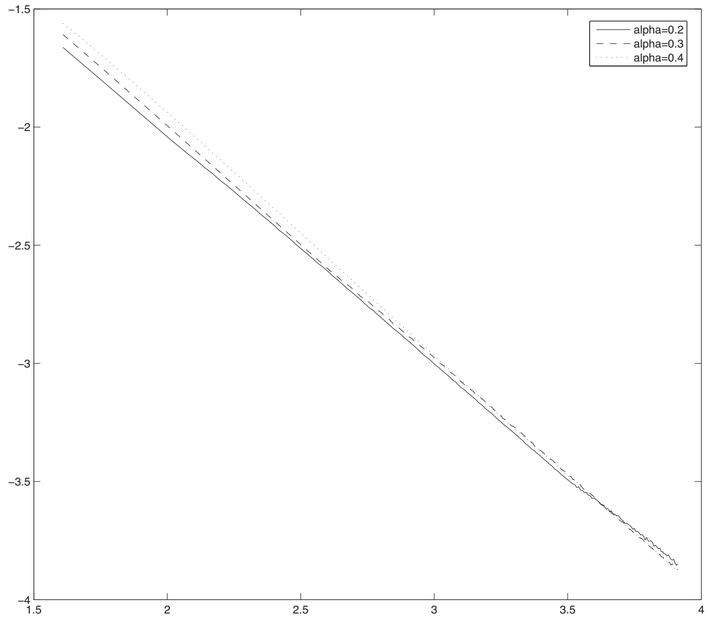

Fig. 5. Plot of $\log\left(\sup_x u_x^\alpha(x,t)\right)$ as a function of $\log t$ for three decreasing values of α, from $t = 5$ until $t = 50$.

Let us now show that (50) is precisely (1). Define

$$u(x,t) := (G^\alpha * v)\,(x,t),$$

so that $\mathcal{H}u = v$ where $\mathcal{H} = \mathrm{Id} - \alpha^2\partial_x^2$ as in (2). It is clear from (48) that $\delta H/\delta v = 1$. Using this in (50) yields

$$v_t = -v_x\left(\partial_x - \alpha^2\partial_x^3\right)^{-1} v_x = -v_x u,$$

which was what was desired. This calculation shows that the regularized equation (1) is Hamiltonian, assuming of course that $\mathfrak{D}$ is a valid Hamiltonian operator. The operator $\mathfrak{D}$ is Hamiltonian if the induced bracket $\{\cdot,\cdot\}$, defined by

$$\{F, G\} = \int_{\mathbb{R}} \frac{\delta F}{\delta v}\mathfrak{D}\frac{\delta G}{\delta v}\,dx, \tag{51}$$

is a Poisson bracket.

Definition 2. A *Poisson bracket* on a manifold M is a skew-symmetric, bilinear operation $\{\,,\}$ on $C^\infty(M)$ satisfying both

1. the Jacobi identity $\{F,\{G,H\}\} + \{H,\{F,G\}\} + \{G,\{H,F\}\} = 0$ and
2. the Leibniz identity $\{FG, H\} = \{F,H\}G + F\{G,H\}$,

for all F, G, and $H \in C^\infty(M)$.

Lemma 1. *The bracket* { , } *induced by* $\mathfrak{D}$ *is skew-symmetric.*

Proof. Because the operator $\mathfrak{L} = \partial_x - \alpha^2\partial_x^3$ has only odd-ordered derivatives, integration by parts gives

$$\langle f, \mathfrak{L}g\rangle = -\langle \mathfrak{L}f, g\rangle,$$

where $\langle\, ,\rangle$ is the natural pairing

$$\langle f, g\rangle = \int_{\mathbb{R}} fg\, dx.$$

Hence we write $\mathfrak{L}^* = -\mathfrak{L}$, which implies $\left(\mathfrak{L}^{-1}\right)^* = -\mathfrak{L}^{-1}$. We use this and the definition of $\mathfrak{D}$ to obtain

$$\begin{aligned}\{F, G\} &= -\int \frac{\delta F}{\delta v} v_x \mathfrak{L}^{-1}\left(v_x \frac{\delta G}{\delta v}\right) dx \\ &= -\int \left(\mathfrak{L}^{-1}\right)^* \left(\frac{\delta F}{\delta v} v_x\right) v_x \frac{\delta G}{\delta v}\, dx \\ &= \int \mathfrak{L}^{-1}\left(v_x \frac{\delta F}{\delta v}\right) v_x \frac{\delta G}{\delta v}\, dx \\ &= -\{G, F\}. \end{aligned}$$

□

Lemma 2. *The bracket* { , } *induced by* $\mathfrak{D}$ *satisfies the Jacobi identity.*

Proof. Directly verifying the Jacobi identity for (51) requires copious amounts of algebra, so we use the multi-vector formalism and Schouten bracket described in [MR99, Chap. 10]. Let us give a sketch of the proof first: using the bi-vector B defined by

$$B = \frac{1}{2}\partial_x \wedge \mathfrak{D}\partial_x, \tag{52}$$

we realize the Poisson bracket as

$$\{F, G\} = \mathbf{i}_B\left(\mathbf{d}F \wedge \mathbf{d}G\right). \tag{53}$$

We will prove that the Schouten bracket of B with itself is zero. Then, by the Jacobi-Schouten identity (see [MR99, Thm. 10.6.2]), we know that the Jacobi identity holds for the bracket { , }. First let us verify (53) by direct computation:

$$\begin{aligned}\mathbf{i}_B\left(\mathbf{d}F \wedge \mathbf{d}G\right) &= \frac{1}{2}\int \frac{\delta F}{\delta v}\mathfrak{D}\frac{\delta G}{\delta v}\, dx - \frac{1}{2}\int \frac{\delta G}{\delta v}\mathfrak{D}\frac{\delta F}{\delta v}\, dx \\ &= \int \frac{\delta F}{\delta v}\mathfrak{D}\frac{\delta G}{\delta v} = \{F, G\},\end{aligned}$$

where we have used skew-symmetry (Lemma 1). Next we use the definition of $\mathfrak{D}$ and $\mathfrak{L} = \partial_x - \alpha^2\partial_x^3$ to write

$$\mathfrak{D}\partial_x = -v_x\mathfrak{L}^{-1}(v_x\partial_x),$$

which implies

$$-v_x^{-1}\mathfrak{L}v_x^{-1}\mathfrak{D}\partial_x = \partial_x.$$

Here we mean simply $v_x^{-1} = 1/v_x$. Using this, we compute the Schouten bracket:

$$\begin{aligned}
[\partial_x \wedge \mathfrak{D}\partial_x, \partial_x \wedge \mathfrak{D}\partial_x] &= [\partial_x \wedge \mathfrak{D}\partial_x, -v_x^{-1}\mathfrak{L}v_x^{-1}\mathfrak{D}\partial_x \wedge \mathfrak{D}\partial_x] \\
&= -\int_{\mathbb{R}} -v_x^{-2}\partial_x(\mathfrak{D}\partial_x) \wedge \mathfrak{L}v_x^{-1}\mathfrak{D}\partial_x \wedge \mathfrak{D}\partial_x \\
&\quad + v_x^{-1}\mathfrak{L}(-v_x^{-2})\partial_x(\mathfrak{D}\partial_x) \wedge \mathfrak{D}\partial_x \wedge \mathfrak{D}\partial_x \, dx.
\end{aligned}$$

The second term vanishes because $\mathfrak{D}\partial_x \wedge \mathfrak{D}\partial_x = 0$. For the first term, we evaluate

$$\begin{aligned}
\mathfrak{L}v_x^{-1}\mathfrak{D}\partial_x &= \partial_x(v_x^{-1}\mathfrak{D}\partial_x) - \alpha^2\partial_x^3(v_x^{-1}\mathfrak{D}\partial_x) \\
&= -v_x^{-2}v_{xx}\mathfrak{D}\partial_x + v_x^{-1}\partial_x(\mathfrak{D}\partial_x) - \alpha^2\partial_x^3(v_x^{-1}\mathfrak{D}\partial_x).
\end{aligned}$$

Since $\mathfrak{D}\partial_x \wedge \mathfrak{D}\partial_x = 0$ and $\partial_x(\mathfrak{D}\partial_x) \wedge \partial_x(\mathfrak{D}\partial_x) = 0$, we are left with

$$[\partial_x \wedge \mathfrak{D}\partial_x, \partial_x \wedge \mathfrak{D}\partial_x] = -\alpha^2 \int_{\mathbb{R}} v_x^{-2}\partial_x(\mathfrak{D}\partial_x) \wedge \partial_x^3(v_x^{-1}\mathfrak{D}\partial_x) \wedge \mathfrak{D}\partial_x \, dx.$$

The only contributions from the ∂_x^3 term that will matter are those that involve either $\partial_x^2(\mathfrak{D}\partial_x)$ or $\partial_x^3(\mathfrak{D}\partial_x)$. With this in mind, we continue the computation:

$$\begin{aligned}
&= -\alpha^2 \int_{\mathbb{R}} v_x^{-2}\partial_x(\mathfrak{D}\partial_x) \wedge (-3v_x^{-2}v_{xx}\partial_x^2(\mathfrak{D}\partial_x) + v_x^{-1}\partial_x^3(\mathfrak{D}\partial_x)) \wedge \mathfrak{D}\partial_x \, dx \\
&= -\alpha^2 \int_{\mathbb{R}} -3v_x^{-4}v_{xx}\partial_x(\mathfrak{D}\partial_x) \wedge \partial_x^2(\mathfrak{D}\partial_x) \wedge (\mathfrak{D}\partial_x) \\
&\quad + v_x^{-3}\partial_x(\mathfrak{D}\partial_x) \wedge \partial_x^3(\mathfrak{D}\partial_x) \wedge (\mathfrak{D}\partial_x) \, dx.
\end{aligned}$$

Integrating the second term by parts to move one derivative *off* the $\partial_x^3(\mathfrak{D}\partial_x)$ term, we find that the entire expression cancels, proving that $[B, B] = 0$ as required. □

Lemma 3. *The bracket* $\{\,,\}$ *induced by* $\mathfrak{D}$ *satisfies the Leibniz identity.*

Proof. We will use the fact that the Leibniz rule holds for functional derivatives:

$$\frac{\delta(FG)}{\delta v} = \frac{\delta F}{\delta v}G + F\frac{\delta G}{\delta v}.$$

The proof of this consists of a simple calculation combined with the observation that $F(v)$ and $G(v)$ do not depend explicitly on x. Using this, we evaluate

$$\begin{aligned}
\{FG, H\}(v) &= \int_{\mathbb{R}} \frac{\delta(FG)}{\delta v}(v)\mathfrak{D}\frac{\delta H}{\delta v}(v)\, dx \\
&= \int_{\mathbb{R}} \frac{\delta F}{\delta v}(v)G(v)\mathfrak{D}\frac{\delta H}{\delta v}(v) + F(v)\frac{\delta G}{\delta v}(v)\mathfrak{D}\frac{\delta H}{\delta v}(v)\, dx \\
&= \{F, H\}(v)G(v) + F(v)\{G, H\}(v),
\end{aligned}$$

where again we have used the fact that $F(v)$ and $G(v)$ are x-independent real numbers. As v was arbitrary, we have shown that the Leibniz identity holds. □

Theorem 3. *The bracket* { , } *induced by* $\mathfrak{D}$ *is a Poisson bracket.*

Proof. By linearity of functional derivatives, it is clear that (51) is bilinear. Then, the preceding lemmas have established that the bracket is skew-symmetric and satisfies the Jacobi and Leibniz identities. □

Casimirs

It happens to be the case that the bracket (51) defined using the operator (49) has no nontrivial Casimirs. Let us quickly verify this. Suppose there exists a function G such that for all F, we have

$$\{F, G\} = 0.$$

By definition (51) of the bracket, this would imply that

$$\int_{\mathbb{R}} \frac{\delta F}{\delta v} \mathfrak{D} \frac{\delta G}{\delta v} \, dx = 0,$$

for all F. The only way this can happen is if in fact

$$\mathfrak{D} \frac{\delta G}{\delta v} = 0.$$

This reads

$$\left(\partial_x - \alpha^2 \partial_x^3\right)^{-1} \left(v_x \frac{\delta G}{\delta v} \right) = 0.$$

Now applying $\left(\partial_x - \alpha^2 \partial_x^3\right)$ to both sides, we obtain

$$\frac{\delta G}{\delta v} = 0,$$

so the only Casimirs are trivial. This is unfortunate—if we had even one nontrivial Casimir, we might use it to prove stability of solutions via the energy-Casimir method. As things stand, deciding the stability of solutions of (1) is likely to be very challenging.

6. Future Directions

At this point, it should be clear that one problem of immediate interest is to prove either that the limit $u = \lim_{\alpha \to 0} u^\alpha$ satisfies the entropy condition, or demonstrate an initial condition v_0 that leads to failure of the entropy condition. Besides this immediate issue, there are several projects of longer-term interest suggested by the current work.

Extension to Higher Dimensions

Consider the vector Burgers equation:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = 0, \tag{54}$$

in either $\mathbb{R}^2$ or $\mathbb{R}^3$. Is it possible that the system

$$\begin{aligned} \mathbf{v}_t + (\mathbf{u} \cdot \nabla)\, \mathbf{v} &= 0, \\ \mathbf{u} - \alpha^2 \Delta \mathbf{u} &= \mathbf{v} \end{aligned}$$

regularizes the vector Burgers equation (54), just as (3) regularizes the scalar Burgers equation?

Extension to One-Dimensional Gas Dynamics

Consider one-dimensional isentropic gas dynamics:

$$\rho_t + (\rho u)_x = 0, \tag{55a}$$

$$(\rho v)_t + \left(\rho u^2 + p\right)_x = 0, \tag{55b}$$

where $p = p(\rho)$. Might it be possible to regularize this system using a mechanism similar to that of (1) ? One candidate system that comes to mind is

$$\rho_t + (\rho u)_x = 0, \tag{56a}$$

$$(\rho u)_t + (\rho u v + p)_x = 0, \tag{56b}$$

$$u - \alpha^2 u_{xx} = v. \tag{56c}$$

Suppose there is zero-α convergence of (ρ^α, u^α) to weak solutions of (55). Then is (56) Hamiltonian in some sense?

Rough Initial Data

It is important to note that both standard and filtered viscosities select the correct entropy solution even in the case of discontinuous initial data, where $u_0 \in L^\infty$ only. This allows one to legitimately use these viscous regularizations to solve Riemann problems. It would be interesting to see whether we can solve Riemann problems directly using (1). That is, what happens to (3) for initial data $v_0 \in L^\infty$?

Other Smoothing Kernels

Another idea is to replace the Helmholtz operator with another operator. For example, we could attempt to regularize Burgers' equation via

$$v_t + u v_x = 0, \tag{57}$$

where u and v are related in any number of ways. One interesting possibility is

$$\widehat{u} = \frac{\widehat{v}}{1 + \alpha|k|}.$$

Now v is, roughly speaking, the "square root" of the Helmholtz operator $\mathcal{H}$ applied to u. Hence u is only *one* derivative smoother than v, whereas in (1), u is *two* derivatives smoother than v. How smooth does u have to be, relative for v, for (57) to genuinely regularize the Burgers equation?

Geometric Structures

Finally, it would be interesting to determine where the Hamiltonian structure (48)–(49) comes from. The Hamiltonian functional (48) is linear in the field variable and therefore does not have the meaning of a kinetic energy. Similarly, what is the meaning of the nonlocal operator in (49)? Is there a Lagrangian structure that yields (1)? Answering these questions will give us physical insight into why our model works the way it does.

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