SYMMETRIC INTEGRALS AND TRIGONOMETRIC SERIES

George E. Cross and Brian S. Thomson

1   Introduction

The first example of a convergent trigonometric series that cannot be expressed in Fourier form is due to Fatou. The series

$$(1) \qquad \sum_{n=1}^{\infty} \frac{\sin nx}{\log(n+1)}$$

converges everywhere to a function that is not Lebesgue, or even Perron, integrable. It follows that the series (1) cannot be represented in Fourier form using the Lebesgue or Perron integrals. In fact this example is part of a whole class of examples as Denjoy [22, pp. 42–44] points out: if $b_n \searrow 0$ and $\sum_{n=1}^{\infty} b_n/n = +\infty$ then the sum of the everywhere convergent series $\sum_{n=1}^{\infty} b_n \sin nx$ is not Perron integrable.

The problem, suggested by these examples, of defining an integral so that the sum function $f(x)$ of the convergent or summable trigonometric series

$$(2) \qquad a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is integrable and so that the coefficients, a_n and b_n, can be written as Fourier coefficients of the function f has received considerable attention in the literature (cf. [10], [11], [21], [22], [26], [32], [34], [37], [44] and [45].) For an excellent survey of the literature prior to 1955 see [27]; [28] is also useful. (In

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some of the earlier works ([10] and [45]) an additional condition on the series conjugate to (2) is imposed.) In addition a secondary literature has evolved devoted to the study of the properties of and the interrelations between the several integrals which have been constructed (for example [3], [4], [5], [6], [7], [8], [12], [13], [14], [15], [16], [17], [18], [19], [23], [29], [30], [31], [36], [38], [39], [40], [42] and [43]).

Excepting for [37] and [32] (which use the approximate symmetric derivative) and [44] (which uses an Abel derivative) all of these integrals are intimately related to the second order symmetric derivative and to the Riemann method of summation (see [46, Vol. I, p. 319]). The solutions of Denjoy [22] and James [26] use explicitly this derivative and introduce a second order integral that recaptures a function from its second symmetric derivative. The solutions of Marcinkiewicz and Zygmund [34] and of Burkill [11] use first order derivatives (the symmetric Borel derivative in the former and the symmetric Cesàro derivative in the latter) and introduce first order integrals as a result; even so the connection with the second order symmetric derivative is immediate and these integrals can be interpreted as a.e. derivatives of the second order integrals of Denjoy and James.

In this article we shall present an account of the Burkill and James integrals from a new point of view. We introduce the notion of symmetric variation and use its properties to define an integral. Some of the technical difficulties that arise in the James integral are simplified by using a device due to Mařík [35] that allows a treatment of the second order integrals as a first order integral. The integral we introduce (in Section 7) is exactly Mařík's version of the James integral but developed directly from the symmetric variation rather than from the standard Perron major/minor function method. This treatment then allows the Burkill (SCP)–integral to be accommodated quite naturally and easily (Section 10).

The applications of this integral to trigonometric series in this article are all achieved by using an integration by parts. In the standard solutions mentioned above the usual proofs involve instead the formal multiplication of trigonometric series, which is a well known technique in this subject ([46, Vol. I, pp. 330–344]). Burkill on the contrary also used an integration by parts argument but it is flawed by the fact that the proof in [11] is incorrect. As later pointed out by H. Burkill [7] the main result in [11] still stands by reverting to the formal multiplication argument. Mařík [35] long ago provided an integration by parts formula for his version of the James integral. Bullen and Mukhopadyay [6] proved a limited version for the (SCP)–integral.

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Finally, thirty years after the appearance of [11], Skljarenko [39] supplied a proof of the integration by parts formula for Burkill's (SCP)–intregral; note that the proof also places this in the setting of the James integral.

We have incorporated some of these ideas here in Sections 3, 11 and 12. Mařík's methods are reproduced and applied in this setting (his notes have not been published to date). They allow, too, a simplification of the proof of the Skljarenko theorem for the (SCP)–integral (see Section 12). The main application to trigonometric series is presented in Section 13; this is Mařík's theorem giving conditions under which a series is the Fourier series of its Riemann sum. As a special case, of course, this returns us to the representation problem with which our discussion began. Section 14 concludes with some classical theorems on trigonometric series that follow as easy corollaries of this work.

2 Some preliminary definitions

The main setting for this article is the real line. All functions are real-valued. The Lebesgue outer measure of a set E of real numbers is denoted as $|E|$. Almost everywhere (a.e.) normally refers to Lebesgue measure; nearly everywhere (n.e.) means excepting a countable set.

The expression

(3) $$\Delta_s^2 F(x, h) = F(x + h) + F(x - h) - 2F(x)$$

is called the *second order symmetric difference* of F at x. Most of our concerns in this article arise from this difference. We recall some of the terminology that has evolved. A function F is said to be *smooth* at a point x if $\Delta_s^2 F(x, h) = o(h)$ as $h \to 0+$. For an arbitrary function F the extreme second order symmetric derivates are defined as

$$\overline{D}_2 F(x) = \limsup_{h \to 0+} \frac{\Delta_s^2 F(x, h)}{h^2} \text{ and } \underline{D}_2 f(x) = \liminf_{h \to 0+} \frac{\Delta_s^2 F(x, h)}{h^2}.$$

If these are equal and are finite we write their common value as $D_2 F(x)$ which is called the *second order symmetric derivative.*

In discussions of integrals the increment of a function F on an interval $[a, b]$ is frequently employed. For symmetric integrals it is often more convenient to employ the expression

(4) $$\lambda_F(a, b) = \lim_{h \to 0+} \left\{ F(b - h) - F(a + h) \right\}$$

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if it exists. This expression has an extra advantage of being symmetric itself (with the $a = b$) and this can be useful.

Since this expression will play an occasional role in the sequel there are several observations that can be made. The proofs are immediate. Note first that if F is continuous on the right at a and on the left at b then

$$(5) \qquad\qquad \lambda_F(a, b) = F(b) - F(a).$$

If F is 2π–periodic and symmetrically continuous, i.e. if

$$\lim_{h \to 0+} \{F(x + h) - F(x - h)\} = 0$$

at each point x, then for any value of a

$$(6) \qquad\qquad \lambda_F(a, a + 2\pi) = 0.$$

If $a < b < c$, if F is symmetrically continuous at b and if both expressions $\lambda_F(a, b)$ and $\lambda_F(b, c)$ exist then

$$(7) \qquad\qquad \lambda_F(a, c) = \lambda_F(a, b) + \lambda_F(b, c).$$

If G is integrable (in the Lebesgue or Denjoy-Perron senses) with an indefinite integral F then $\lambda_G(a, b)$ can be directly obtained from F. For an arbitrary function F defined on an interval $[a, b]$ we shall write

$$(8) \qquad \Lambda_F(a, b) = \lim_{h \to 0+} \frac{F(a) - F(a + h) - F(b - h) + F(b)}{h}.$$

If $F(x) = \int_a^x G(t)\, dt$ then

$$\frac{F(a) - F(a + h) - F(b - h) + F(b)}{h} - \lambda_G(a, b)$$

$$= h^{-1} \int_0^h \{G(b - t) - G(a + t) - \lambda_G(a, b)\}\, dt$$

and if $\lambda_G(a, b)$ exists the integrand tends to zero. Thus we have proved that if $F = \int G$ and $\lambda_G(a, b)$ exists then

$$(9) \qquad\qquad \lambda_G(a, b) = \Lambda_F(a, b).$$

Like (4) the expression (8) has an extra advantage of being symmetric itself (with $a = b$ the numerator is a second symmetric difference). It is

$\Lambda_F(a, b)$ we shall use more frequently than $\lambda_G(a, b)$ and our attention now turns to a development of its more elementary properties.

The expression is additive in the sense that

(10) $$\Lambda_{F+G}(a, b) = \Lambda_F(a, b) + \Lambda_G(a, b)$$

if both exist. If F is linear then $\Lambda_F(a, b) = 0$; moreover in that case it follows from (10) that $\Lambda_{F+G} = \Lambda_G$ for any other function G. If $F - G$ is linear and either of the two expressions exist then $\Lambda_F(a, b) = \Lambda_G(a, b)$. If the one-sided derivatives $F'_+(a)$ and $F'_-(b)$ exist for a function F then

(11) $$\Lambda_F(a, b) = F'_-(b) - F'_+(a).$$

In particular then, if F is convex on an open interval that contains the points a and b, the expression $\Lambda_F(a, b)$ must exist; we shall use this fact in Definition 6 below. If F is smooth and 2π–periodic then for any value of a

(12) $$\Lambda_F(a, a + 2\pi) = 0.$$

If $a < b < c$, if F is smooth at b and both expressions $\Lambda_F(a, b)$ and $\Lambda_F(b, c)$ exist then

(13) $$\Lambda_F(a, c) = \Lambda_F(a, b) + \Lambda_F(b, c).$$

Finally here is one last computation (from [35, (112), p. 65]) for the expression (8). Suppose that F and G are continuous on an interval $[a, b]$, that $\Lambda_F(a, b)$ exists, that $G(a) = G(b)$ and that both derivatives $G'_-(b)$ and $G'_+(a)$ exist. Then

(14) $$\Lambda_{FG}(a, b) = G(a)\Lambda_F(a, b) + \left(F(b)G'_-(b) - F(a)G'_+(a) \right).$$

To check (14) write $G(a) = G(b) = \beta$,

$$L(x) = F(a) + (F(b) - F(a)(x - a)/(b - a),$$

$F_1 = F - L$, $G_1 = G - \beta$. Then $\Lambda_{F_1 G_1}(a, b) = 0$, $\Lambda_{\beta F_1}(a, b) = \beta\Lambda_F(a, b)$ and

$$\Lambda_{LG}(a, b) = \left(F(b)G'_-(b) - F(a)G'_+(a) \right).$$

Finally then (14) follows from

$$\Lambda_{FG}(a, b) = \Lambda_{F_1 G_1}(a, b) + \Lambda_{\beta F_1}(a, b) + \Lambda_{LG}(a, b).$$

3   Mařík's symmetric difference

Our study will require some attention to a symmetric difference introduced by Mařík [35]. This difference arises from the observation that if an integrable function F has a symmetric derivative at a point then the indefinite integral of F has a second order symmetric derivative at that point and the values are the same. The Mařík difference is a measure of this fact and will prove useful in a discussion of the integration by parts formula in Section 11. This material is all due to Mařík [35] and reproduced here as it has not been published.

1 DEFINITION. Let F be integrable and defined everywhere in a neighbourhood of a point x. Then, for $h > 0$, we write

$$\mathrm{M}_s^2 F(x, h) = \frac{F(x + h) - F(x - h)}{2h} - \frac{1}{h^2} \int_0^h \{F(x + t) - F(x - t)\} \, dt$$

whenever this makes sense.

LEMMA 2 *Given any x and a positive η, suppose that $|F''(\tau)| < w$ for all $\tau \in (x - \eta, x + \eta)$. Then $|\mathrm{M}_s^2 F(x, h)| \leq hw/6$ for all $0 < h < \eta$.*

PROOF. (Reproduced from [35, (110), p .64].) Write $g(t) = F(x+t) - F(x-t)$ and $G(t) = tg(t) - 2 \int_0^t g(s) \, ds$ $(0 < t < \eta)$. The following computations are immediate: $G'(t) = tg'(t) - g(t)$, $G''(t) = tg''(t)$, $|G''(t)| \leq 2tw$, $G(0) = G'(0) = 0$, $|G'(t)| \leq wt^2$ and $|G(t)| \leq wt^3/3$. From these

$$|\mathrm{M}_s^2 F(x, h)| = \frac{1}{2h^2} |G(h)| \leq \frac{hw}{6}$$

for all $0 < h < \eta$ now follows.

LEMMA 3 *Suppose that F has a finite symmetric derivative at a point x. Then $\mathrm{M}_s^2 F(x, h) \to 0$ as $h \to 0+$.*

PROOF. (Reproduced from [35, (111), p .65].) Let α denote the symmetric derivative of F at x. Then $F(x+t) - F(x-t) = 2\alpha t + F_1(t)$ where $F_1(t) = o(t)$ as $t \to 0+$. Then

$$\mathrm{M}_s^2 F(x, h) = \alpha + \frac{F_1(h)}{2h} - \frac{1}{h^2} \int_0^h (2\alpha t + F_1(t)) \, dt$$

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$$= \frac{F_1(h)}{2h} - \frac{1}{h^2} \int_0^h F_1(t)\, dt$$

and this is evidently tending to zero as $h \to 0+$.

LEMMA 4 *Let α, β and γ be nonnegative real numbers. Let $\{\alpha_n\}$ and $\{\xi_n\}$ be sequences of real numbers with $|\alpha_n| \leq \alpha$. Define*

$$F(x) = \sum_{n=1}^{\infty} \frac{\alpha_n}{n^2} f(\xi_n + nx)$$

where f is a real function satisfying $|f(x)| \leq \beta$ and $|f''(x)| \leq \gamma$ for all x. Then $|\mathrm{M}_s^2 F(x, h)| \leq 2\alpha\sqrt{\beta\gamma}$.

PROOF. (Reproduced from [35, (120), p .70].) We may assume that each of α, β and γ is positive. Write $g_n(t) = f(\xi_n + nx + t) - f(\xi_n + nx - t)$ and $K_n(x) = f(\xi_n + x)$. If we define

$$B_n(t) = \frac{g_n(nt)}{2t} - \frac{1}{t^2} \int_0^t g_n(n\tau)\, d\tau$$

then we can easily establish the identity

(15) $$\mathrm{M}_s^2 F(x, h) = \sum_{n=1}^{\infty} \frac{\alpha_n}{n^2} B_n(h)$$

that will allow us to establish the estimate required in the lemma.

Since $|f| \leq \beta$, $|g_n| \leq 2\beta$ and hence $|B_n(h)| \leq 3\beta h^{-1}$. We shall need as well the estimate $|B_n(h)| \leq n^2 h\gamma/6$. To obtain this we write

$$B_n(h) = n\left(\frac{g_n(nh)}{2nh} - \frac{1}{n^2h^2}\int_0^{nh} g_n(\tau)\, d\tau\right)$$

$$= n\mathrm{M}_s^2 K_n(nx, nh).$$

Now $|f''| \leq \gamma$ and so also $|K_n''| \leq \gamma$. Thus, using Lemma 2, we have

$$|B_n(h)| = n\,|\mathrm{M}_s^2 K_n(nx, nh)| \leq \frac{n^2 h\gamma}{6}$$

as we wished to prove.

Define $t_0 = 3\sqrt{\beta/\gamma}$ (i.e. so that $\gamma/3 = 3\beta/t_0^2$) and define the function

$$\phi(t) = \begin{cases} \gamma/3 & \text{for } 0 < t < t_0, \\ 3\beta/t^2 & \text{for } t_0 \leq t. \end{cases}$$

Note that ϕ is continuous and nonincreasing and that, because of the inequalities $|B_n(h)| \leq 3\beta h^{-1}$ and $|B_n(h)| \leq n^2 h\gamma/6$, we have

$$|B_n(h)| \leq hn^2\phi(nh).$$

Combining this with (15) we obtain

$$|\mathrm{M}_s^2 F(x,h)| \leq \alpha \sum_{n=1}^{\infty} h\phi(nh).$$

To get an upper estimate on this sum note that

$$\int_{(n-1)h}^{nh} \phi(\tau)\,d\tau \geq h\phi(nh)$$

and so

$$\sum_{n=1}^{\infty} h\phi(nh) \leq \int_0^{\infty} \phi(\tau)\,d\tau = \frac{\gamma t_0}{3} + \frac{3\beta}{t_0} = \frac{2\gamma t_0}{3} = 2\sqrt{\beta\gamma}.$$

So finally we have $|\mathrm{M}_s^2 F(x,h)| \leq 2\alpha\sqrt{\beta\gamma}$ as required.

THEOREM 5 *Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences of real numbers and suppose that*

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \{a_n \cos nx + b_n \sin nx\}.$$

Then F is continuous, 2π–periodic and

$$|\mathrm{M}_s^2 F(x,h)| \leq 2 \sup_n \sqrt{a_n^2 + b_n^2}$$

for all x and all $h > 0$.

PROOF. (Reproduced from [35, (121),p .71].) Write $\alpha_n = \sqrt{a_n^2 + b_n^2}$, $f(x) = \sin x$ and $a_n \cos nx + b_n \sin nx = \alpha_n \sin(\xi_n + nx)$. The theorem now follows directly from Lemma 4.

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4   Basic definitions

The relation $D_2 F(x) = f(x)$ for x in a set E can be written as the requirement that for every $\epsilon > 0$ there is a $\delta(x) > 0$ so that

$$\left| F(x+t) + F(x-t) - 2F(x) - f(x)t^2 \right| < \epsilon t^2$$

for $x \in E$ and $0 < t < \delta(x)$. This observation taken in concert with the variational ideas used in nonabsolute integration on the real line suggests the following definition. We express it for a general function of pairs $\xi(x, h)$ but it will be applied mainly in situations where

$$\xi(x, h) = F(x+h) + F(x-h) - 2F(x)$$

or

$$\xi(x, h) = F(x+h) + F(x-h) - 2F(x) - f(x)h^2$$

for some real functions F and f.

6 DEFINITION. Let E be a bounded set with $a = \inf E$, $b = \sup E$ and let ξ be a function defined for pairs (x, h) with $x \in E$ and h sufficiently small. Then we write

(16) $$V_s^2(\xi, E) = \inf \Lambda_G(a, b) = \inf \left\{ G'_-(b) - G'_+(a) \right\}$$

where the infimum is taken over all functions G convex on an open interval that contains $\overline{E}$ such that for every $x \in E$ there is a positive number $\delta = \delta(G, x)$ so that
(17) $$|\xi(x, h)| < \Delta_s^2 G(x, h)$$

for all $0 < h < \delta$.

Certainly $0 \leq V_s^2(\xi, E) \leq +\infty$. If in the definition no such functions G exist then of course we take $V_s^2(\xi, E) = +\infty$. The strict inequality in (17) may obviously be replaced by a weaker nonstrict one. Note that because G is convex on an interval that includes the points a and b, by (11), the expression $\Lambda_G(a, b)$ must exist. We call $V_s^2(\xi, E)$ the *second symmetric variation* of the function ξ. Thus far it is defined for all bounded sets E; for unbounded E we can simply take

$$V_s^2(\xi, E) = \lim_{n \to \infty} V_s^2 \left(\xi, E \cap (-n, n) \right)$$

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but this will play no particular role in the sequel except to permit the variation to be defined on all sets of real numbers. The variation in Definition 6 is defined relative to the containing interval $[a, b]$; notice that nothing is changed by using any larger interval that contains E. To see this observe that the expression for $V_s^2(\xi, E)$ would only be larger if $[a, b]$ were replaced by some larger interval in (16). But on the other hand if a function G is given convex on an open interval containing $[a, b]$ then it may be extended to be convex on an open interval containing any larger interval $[c, d]$ in such a way that $\Lambda_G(a, b) = \Lambda_G(c, d)$; thus $V_s^2(\xi, E)$ is made no larger by employing larger intervals. Thus in the sequel we may take any containing interval to use in (16).

Our first three lemmas offer useful tests for zero variation.

LEMMA 7 *Let E be a set of real numbers having measure zero and suppose that ξ is a real-valued function defined for pairs (x, h) with $x \in E$ and h sufficiently small. If $\xi(x, h) = O(h^2)$ as $h \to 0+$ for every $x \in E$ then $V_s^2(\xi, E) = 0$.*

PROOF. We may suppose that E is bounded, say that $\overline{E} \subset (a, b)$. Let $\epsilon > 0$ and let N be a $\mathcal{G}_\delta$ set of measure zero containing E. There is (cf. [2, p. 124]) a positive, absolutely continuous, increasing function g so that $g'(x) = +\infty$ at each point of N and so that $|g(x)| < \epsilon$ on $[a, b]$. Let G be an indefinite integral of g. Then G is convex, $\Lambda_G(a, b) < \epsilon$ and $|\xi(x, h)| < \Delta_s^2 G(x, h)$ for all $x \in E$ and sufficiently small h. This is clear since $\Delta_s^2 G(x, h)/h^2 \to +\infty$ and $\xi(x, h) = O(h^2)$ as $h \to 0+$. From this it follows that

$$V_s^2(\xi, E) \leq \Lambda_G(a, b) < \epsilon$$

and the lemma follows.

LEMMA 8 *Let E be a set of real numbers and suppose that ξ is a real-valued function defined for pairs (x, h) with $x \in E$ and h sufficiently small. If $\xi(x, h) = o(h^2)$ as $h \to 0+$ for every $x \in E$ then $V_s^2(\xi, E) = 0$.*

PROOF. Again we may suppose that E is bounded, say that $\overline{E} \subset (a, b)$. Let $\epsilon > 0$ and write $G(x) = \epsilon x^2$. Note that $\Delta_s^2 G(x, h) = 2\epsilon h^2$ and that $\Lambda_G(a, b) = 2\epsilon(b - a)$. As $\xi(x, h) = o(h^2)$ as $h \to 0+$ for every $x \in E$ we

have $|\xi(x, h)| < \Delta_s^2 G(x, h)$ again for every $x \in E$ and sufficiently small h. By definition then

$$\text{V}_s^2(\xi, E) \leq \Lambda_G(a, b) = 2\epsilon(b - a).$$

The lemma follows since ϵ is arbitrary.

LEMMA 9 *Let C be a countable set of real numbers and suppose that ξ is a real-valued function defined for pairs (x, h) with $x \in C$ and h sufficiently small. If $\xi(x, h) = o(h)$ as $h \to 0+$ for every $x \in C$ then $\text{V}_s^2(\xi, C) = 0$.*

PROOF. As before we may suppose that C is bounded, say that $\overline{C} \subset (a, b)$. Let $\epsilon > 0$. Let $\{c_1, c_2, c_3, \ldots\}$ be an enumeration of C. There must be a number $\delta(c_i) > 0$ so that

(18) $$|\xi(c_i, h)| < \epsilon 2^{-i} h$$

for $0 < h < \delta(c_i)$. Define $G_i(x) = 0$ ($x \leq c_i$) and $G_i(x) = \epsilon 2^{-i}(x - c_i)$ ($x > c_i$). Then each G_i is convex, $\Lambda_{G_i}(a, b) = \epsilon 2^{-i}$ and, by (18),

(19) $$|\xi(c_i, h)| < \epsilon 2^{-i} h \leq \Delta_s^2 G_i(c_i, h)$$

for sufficiently small h.

Now simply write $G = \sum_{n=1}^{\infty} G_i$. G is convex, $\Lambda_G(a, b) \leq \epsilon$ and, by (19),

$$|\xi(c_i, h)| < \epsilon 2^{-i} h \leq \Delta_s^2 G(c_i, h)$$

for each $c_i \in C$ and sufficiently small h. By definition then

$$\text{V}_s^2(\xi, E) \leq \Lambda_G(a, b) \leq \epsilon$$

and the lemma follows since ϵ is arbitrary.

Let us show now that the variation is subadditive both as a function of sets and as a functional.

LEMMA 10 *Let E be a set of real numbers and suppose that ξ_1 and ξ_2 are real-valued functions defined for pairs (x, h) with $x \in E$ and h sufficiently small. Then*

$$\text{V}_s^2(\xi_1 + \xi_2, E) \leq \text{V}_s^2(\xi_1, E) + \text{V}_s^2(\xi_2, E).$$

PROOF. If $|\xi_1(x, h)| < \Delta_s^2 G_1(x, h)$ for all $0 < h < \delta_1(x)$ and $|\xi_2(x, h)| < \Delta_s^2 G_2(x, h)$ for all $0 < h < \delta_2(x)$ then

$$|\xi_1(x, h) + \xi_2(x, h)| < \Delta_s^2 G_1(x, h) + \Delta_s^2 G_2(x, h)$$

for all $0 < h < \min\{\delta_1(x), \delta_2(x)\}$. As

$$\Lambda_{G_1+G_2}(a, b) = \Lambda_{G_1}(a, b) + \Lambda_{G_2}(a, b)$$

when these latter both exist the lemma must follow.

LEMMA 11 *Let $\{E_i\}$ be a sequence of sets and suppose that $E \subset \bigcup_{i=1}^{\infty} E_i$. Suppose that ξ is a function defined for pairs (x, h) with $x \in \bigcup_{i=1}^{\infty} E_i$ and h sufficiently small. Then*

$$V_s^2(\xi, E) \leq \sum_{i=1}^{\infty} V_s^2(\xi, E_i).$$

PROOF. We may suppose that the closures of all the sets are contained entirely within a single interval (a, b), that the sets $\{E_i\}$ are disjoint, that each $V_s^2(\xi, E_i) < +\infty$ and that $\sum_{i=1}^{\infty} V_s^2(\xi, E_i)$ converges. The general case follows routinely once this situation is handled. Choose functions G_i convex on $[a, b]$ such that for every $x \in E_i$ there is a $\delta(x) > 0$ (depending on G_i) so that $|\xi(x, h)| < \Delta_s^2 G_i(x, h)$ for all $0 < h < \delta(x)$ and so that

$$0 \leq \Lambda_{G_i}(a, b) < V_s^2(\xi, E_i) + \epsilon 2^{-i}.$$

We can insist that $G_i(a) = G_i'(a) = 0$, and that

$$G_i(x) = \int_a^x g_i(t)\, dt$$

where g_i is nonnegative and increasing on $[a, b]$, $g_i(a) = 0$ and $g_i(b) = \Lambda_{G_i}(a, b)$. Because

$$\sum_{i=1}^{\infty} V_s^2(\xi, E_i) + \epsilon 2^{-i} < +\infty$$

we can write $g(x) = \sum_{i=1}^{\infty} g_i(x)$ which is uniformly convergent on $[a, b]$. So also $G(x) = \sum_{i=1}^{\infty} G_i(x) = \int_a^x g(t)\, dt$ is continuous and convex on $[a, b]$, and

$$\Lambda_G(a, b) \leq g(b) - g(a) = \sum_{i=1}^{\infty} g_i(b) = \sum_{i=1}^{\infty} \Lambda_{G_i}(a, b) < +\infty.$$

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If $x \in E$ then $x \in E_i$ for some i so that $|\xi(x,h)| < \Delta_s^2 G_i(x,h)$ for all $0 < h < \delta(x)$. Write $H = \sum_{j \neq i} G_j$. H is convex and so

$$|\xi(x,h)| < \Delta_s^2 G_i(x,h) + \Delta_s^2 H(x,h) = \Delta_s^2 G(x,h)$$

for all $0 < h < \delta(x)$. Consequently, by definition,

$$V_s^2(\xi, E) \leq \Lambda_G(a,b) = \sum_{i=1}^{\infty} \Lambda_{G_i}(a,b) \leq \sum_{i=1}^{\infty} V_s^2(\xi, E_i) + \epsilon.$$

Letting $\epsilon \to 0+$ completes the proof.

The variation immediately provides an equivalence relation that is central to our theory. We refer to this as "variational equivalence" and it shall play the same role in the integration theory that the similarly named concept plays in the Henstock theory for the Denjoy-Perron integral (as in, for example, [25]).

12 DEFINITION. Let E be a set of real numbers and let ξ_1 and ξ_2 be functions defined for pairs (x,h) with $x \in E$ and h sufficiently small. We write $\xi_1 \equiv \xi_2$ in E if $V_s^2(\xi_1 - \xi_2, E) = 0$.

It is easy to check that this is an equivalence relation and that if $\xi_1 \equiv \xi_2$ in E then also $\xi + \xi_1 \equiv \xi + \xi_2$ in E for any other function ξ.

The variational expression $V_s^2(\xi, E)$ when viewed as a function of the set E for ξ fixed provides an outer measure that will also prove central in our theory.

13 DEFINITION. Let E be a set of real numbers and let ξ be a function defined for pairs (x,h) with $x \in E$ and h sufficiently small. We write

$$\xi^*(E) = V_s^2(\xi, E).$$

The main theorems for the variational measure follow. The first asserts that the equivalence relation preserves the measure and the second that the measure is a true outer measure.

THEOREM 14 *Let ξ_1 and ξ_2 be functions defined for pairs (x,h) with $x \in E$ and h sufficiently small. If $\xi_1 \equiv \xi_2$ in E then $\xi_1^*(E) = \xi_2^*(E)$.*

PROOF. By Lemma 10

$$\mathrm{V}_s^2(\xi_1, E) \le \mathrm{V}_s^2(\xi_1 - \xi_2, E) + \mathrm{V}_s^2(\xi_2, E)$$

and so $\xi_1^*(E) \le \xi_2^*(E)$. Since $\xi_2^*(E) \le \xi_1^*(E)$ may be similarly proved the theorem follows.

THEOREM 15 *Let ξ be a function defined for pairs (x, h) with x real and h sufficiently small. Then ξ^* is an outer measure on the real line.*

PROOF. This follows directly from Lemma 11.

5 Properties of the second symmetric variation for real functions

We shall use the following convenient notation. If F, G and f are finite functions then we define the functions $\Delta_s^2 F$, $f\Delta_s^2 G$, $f(\Delta\ell)^2$ as the expressions

$$\Delta_s^2 F(x, h) = F(x + h) + F(x - h) - 2F(x),$$

$$f\Delta_s^2 G(x, h) = f(x)(G(x + h) + G(x - h) - 2G(x))$$

and

$$f(\Delta\ell)^2(x, h) = f(x)h^2$$

wherever these are defined.

In particular then a meaning is now attached to the equivalence relations $\Delta_s^2 F \equiv f\Delta_s^2 G$, $f(\Delta\ell)^2 \equiv \Delta_s^2 F$, etc. and this relation supplies a convenient way of expressing the variational ideas of the integration theory. The variational measures $\Delta_s^2 F^*(E) = \mathrm{V}_s^2(\Delta_s^2 F, E)$ play a key role too.

All of the technical properties of the second order symmetric integrals that we shall study are expressed in properties of this equivalence relation and the measures. The theorems of this section provide a systematic account of the properties of the equivalence relation; properties of the measures are discussed in the next section.

Throughout the theorems in this section we assume that H, F, and G denote continuous functions on an interval $[a, b]$ and f, g denote arbitrary finite functions defined on an interval (a, b).

14

THEOREM 16 *If $\Delta_s^2 H \equiv 0$ in (a, b) then H is linear in (a, b).*

PROOF. Let $H_1 = H - L$ where L is linear and $H_1(a) = H_1(b) = 0$. Then $\Delta_s^2 H_1 = \Delta_s^2 H$ and so $\Delta_s^2 H_1 \equiv 0$. Now we must show that H_1 vanishes in (a, b). Let $\epsilon > 0$. Choose a function G convex on an open interval that contains $[a, b]$ and a positive function δ on (a, b) so that $\Lambda_G(a, b) < \epsilon$ and so that

(20) $$\left| \Delta_s^2 H_1(x, h) \right| < \Delta_s^2 G(x, h) \quad (a < x < b)$$

for all $0 < h < \delta(x)$. By subtracting from G a linear function we may suppose that $G(a) = G(b) = 0$. Note that, because $\Lambda_G(a, b) < \epsilon$, this requires

(21) $$0 \geq G(x) > -\epsilon(b - a) \quad (a < x < b).$$

From (20) and the continuity of G and H we see that both functions $G + H_1$ and $G - H_1$ are convex on (a, b) and consequently from (21) we have $|H_1(x)| < \epsilon(b - a)$ for all $a < x < b$. Since ϵ is arbitrary H_1 must vanish as stated.

COROLLARY 17 *If $\Delta_s^2 F \equiv \Delta_s^2 G$ in (a, b) then $F - G$ is linear in (a, b).*

PROOF. This is immediate.

COROLLARY 18 *If $\Delta_s^2 F \equiv \Delta_s^2 G$ in (a, b) and either of $\Lambda_F(a, b)$ and $\Lambda_G(a, b)$ exists then $\Lambda_F(a, b) = \Lambda_G(a, b)$.*

PROOF. This is immediate from the theorem and the observation (stated in Section 2) that if $F - G$ is linear and either expression exists then $\Lambda_F(a, b) = \Lambda_G(a, b)$.

THEOREM 19 *If $\Delta_s^2 F \equiv f(\Delta \ell)^2$ in a set E then F is smooth at every point in E and $D_2 F(x) = f(x)$ almost everywhere in E.*

PROOF. We may suppose that E is bounded, say that $\overline{E} \subset (a, b)$. Let $\epsilon > 0$ and choose a function G convex on an open interval containing $[a, b]$ so that $\Lambda_G(a, b) < \epsilon$ and such that for every $x \in E$ there is a $\delta(x) > 0$ so that

(22) $$\left| \Delta_s^2 F(x, h) - f(x)h^2 \right| < \Delta_s^2 G(x, h)$$

for all $0 < h < \delta(x)$. For any $a < x < b$ and for small enough h we have

(23) $\qquad \Delta_s^2 G(x,h) \leq h \left(G'_-(x+h) - G'_+(x-h) \right) < h\Lambda_G(a,b) < \epsilon h.$

Thus from (22) and (23) we see that

$$\left| \Delta_s^2 F(x,h) \right| \leq |f(x)|h^2 + \epsilon h$$

and from this it follows that F is smooth at each point of E.

For the second part of the proof let E_0 denote the set of points x in E at which the statement $\mathrm{D}_2 \, F(x) = f(x)$ fails. For each such point x there must exist a $\eta(x) > 0$ and a sequence $h_i \to 0+$ (also depending on x) so that

(24) $\qquad \left| \Delta_s^2 F(x,h_i) - f(x)h_i{}^2 \right| > \eta(x)h_i{}^2.$

Let $c > 0$ and define $E_c = \{x \in E_0; \, \eta(x) > c\}$. We show that each such set E_c has measure zero and it must follow that E_0 has measure zero too so that the theorem is proved.

Suppose not. Then $|E_c| > 0$ for some fixed $c > 0$. The collection of intervals of the form $[x - h, x + h] \subset (a,b)$ with $x \in E_c$, $0 < h < \delta(x)$ and

(25) $\qquad \left| \Delta_s^2 F(x,h) - f(x)h^2 \right| > ch^2.$

is, by (24), a Vitali cover of E_c. Select a finite disjoint sequence of such intervals $\{[x_i - h_i, x_i + h_i]\}$ $(i = 1, 2, \ldots, n)$ so that

$$|E_c| < 4 \sum_{i=1}^{n} h_i.$$

Then, using (22) and (25), we have

$$|E_c| < 4c^{-1} \sum_{i=1}^{n} \left| \Delta_s^2 F(x_i, h_i) - f(x_i)h_i{}^2 \right| / h_i$$

$$< 4c^{-1} \sum_{i=1}^{n} \left| \Delta_s^2 G(x_i, h_i) \right| / h_i$$

$$< 4c^{-1} \sum_{i=1}^{n} (G'_-(x_i + h_i) - G'_+(x_i - h_i))$$

$$< 4c^{-1}\Lambda_G(a,b) < 4c^{-1}\epsilon.$$

As ϵ is arbitrary this is a contradiction and the theorem is proved.

COROLLARY 20 *If $\Delta_s^2 F \equiv f(\Delta \ell)^2$ in (a, b) then f is measurable.*

PROOF. By the theorem f is almost everywhere the second symmetric derivative of a continuous function and so f is measurable.

COROLLARY 21 *If $f(\Delta \ell)^2 \equiv 0$ in a set E then $f = 0$ almost everywhere in E.*

PROOF. By the theorem f is almost everywhere in E the second symmetric derivative of any constant function and so f must vanish almost everywhere in E.

COROLLARY 22 *If $f(\Delta \ell)^2 \equiv g(\Delta \ell)^2$ in a set E then $f = g$ almost everywhere in E.*

PROOF. This is immediate from the preceding corollary.

THEOREM 23 *If $D_2 F(x) = f(x)$ everywhere in a set E then $\Delta_s^2 F \equiv f(\Delta \ell)^2$ in E.*

PROOF. The relation $D_2 F(x) = f(x)$ for x in a set E is exactly equivalent to the requirement $F(x + t) + F(x - t) - 2F(x) - f(x)t^2 = o(t^2)$ for $x \in E$ as $t \to 0+$. The theorem now follows directly from Lemma 8.

THEOREM 24 *If $f = 0$ almost everywhere in a set E then $f(\Delta \ell)^2 \equiv 0$ in E.*

PROOF. This follows directly from Lemma 7.

COROLLARY 25 *If $f = g$ almost everywhere in a set E then $f(\Delta \ell)^2 \equiv g(\Delta \ell)^2$ in E.*

PROOF. This is immediate from the theorem.

THEOREM 26 *If f is Lebesgue or Denjoy-Perron integrable in $[a, b]$ with a second indefinite integral F then $\Delta_s^2 F \equiv f(\Delta \ell)^2$ in (a, b).*

PROOF. We may suppose that f is Denjoy-Perron integrable on $[a,b]$, that H is an indefinite integral of f and that $F(x) = \int_a^x H(t)\,dt$. Let $\epsilon > 0$.

By a well-known characterization of the Denjoy-Perron integral (eg. [25, pp. 225–233]) there must exist a continuous, positive, increasing function g on $[a,b]$ with $g(b) - g(a) < \epsilon$ and a $\delta(x) > 0$ $(a < x < b)$ so that

(26) $$|H(x+t) - H(x) - f(x)t| < |g(x+t) - g(x)|$$

for all $x \in (a,b)$ and for all $0 < |t| < \delta(x)$. We may set $g(x) = g(a)$ $(x < a)$ and $g(x) = g(b)$ $(x > b)$. Write $G(x) = \int_a^x g(t)\,dt$, fix x and fix $h \in (0, \delta(x))$. We integrate the expression (26) for t in the intervals $[0, h]$ and $[-h, 0]$ and subtract the results. This gives immediately

$$\left| \Delta_s^2 F(x,h) - f(x)h^2 \right| < \Delta_s^2 G(x,h)$$

for all $0 < h < \delta(x)$.

Note that G is convex and that $\Lambda_G(a,b) = g(b) - g(a) < \epsilon$. If we compare with Definition 6 we see that we have proved that

$$\mathrm{V}_s^2(\Delta_s^2 F - f(\Delta\ell)^2, (a,b)) < \epsilon.$$

From this we evidently obtain that $\Delta_s^2 F \equiv f(\Delta\ell)^2$ in (a,b) as required.

THEOREM 27 *If $f \geq 0$ and $\Delta_s^2 F \equiv f(\Delta\ell)^2$ in (a,b) for a function F continuous on $[a,b]$ then f is Lebesgue integrable and F is a second indefinite Lebesgue integral for f.*

PROOF. Let $\epsilon > 0$ and choose functions G_n convex on an open interval containing $[a,b]$ so that $\Lambda_{G_n}(a,b) < 1/n$, so that $G_n(a) = G_n(b) = 0$ and such that for every $x \in (a,b)$ there is a $\delta_n(x) > 0$ so that

$$\left| \Delta_s^2 F(x,h) - f(x)h^2 \right| < \Delta_s^2 G_n(x,h)$$

for all $0 < h < \delta_n(x)$. Because f is nonnegative this shows that

$$\Delta_s^2 G_n(x,h) + \Delta_s^2 F(x,h) > 0 \quad (a < x < b)$$

for small enough h. Consequently $F + G_n$ is convex. The limit is convex too and so we obtain that F itself is convex. Since F is convex it is the integral of some monotonic function F_1, $F_1' = f_1$ exists almost everywhere

and f_1 is Lebesgue integrable. This means then that the second symmetric derivative of F exists and is equal to f_1 almost everywhere. By Theorem 19, $f = f_1$ almost everywhere so, since f_1 is Lebesgue integrable, f is Lebesgue integrable. Let H be a second indefinite integral for f. By Theorem 26 then $\Delta_s^2 H \equiv f(\Delta \ell)^2$ in (a, b). Since we already have $\Delta_s^2 F \equiv f(\Delta \ell)^2$ in (a, b) this means that $\Delta_s^2 F \equiv \Delta_s^2 H$ in (a, b) and so, by Corollary 17, H and F differ by a linear function. The theorem follows.

6   Measure properties

This section is devoted mostly to the properties of the variational measures and especially to the real-variable properties of the function F as expressed by the measure $\Delta_s^2 F^*$. This measure expresses smoothness properties of F and various properties of the second symmetric derivative of F.

THEOREM 28 *A function F is smooth at a point x_0 if and only if*

$$\Delta_s^2 F^*(\{x_0\}) = 0.$$

PROOF. By Theorem 19 if $\Delta_s^2 F \equiv 0$ in the set $\{x_0\}$ then F is smooth at x_0. The converse follows directly from Lemma 9.

THEOREM 29 *If F is convex on an open interval containing $[a, b]$ then*

$$\Delta_s^2 F^* ((a, b)) = \Lambda_F(a, b).$$

PROOF. If G is convex on an open interval containing $[a, b]$ and

$$\left| \Delta_s^2 F(x, h) \right| < \Delta_s^2 G(x, h) \quad (a < x < b)$$

for sufficiently small h then evidently $G - F$ is convex on (a, b). Consequently, by (10),
$$0 \le \Lambda_{G-F}(a, b) = \Lambda_G(a, b) - \Lambda_F(a, b)$$
and so $\Lambda_F(a, b) \le \Lambda_G(a, b)$. From this it follows that

(27) $$\mathrm{V}_s^2(\Delta_s^2 F, (a, b)) \ge \Lambda_F(a, b).$$

On the other hand take $G(x) = F(x) + \epsilon x^2$ for any $\epsilon > 0$. Certainly $|\Delta_s^2 F(x,h)| < \Delta_s^2 G(x,h)$ $(a < x < b)$ for sufficiently small h and G itself is convex. From this it follows that

$$(28) \qquad V_s^2(\Delta_s^2 F, (a,b)) \leq \Lambda_G(a,b) = \Lambda_F(a,b) + 2\epsilon(b-a).$$

As ϵ is arbitrary (27) and (28) together show that $V_s^2(\Delta_s^2 F, (a,b)) = \Lambda_F(a,b)$ as required.

COROLLARY 30 *If $F(x) = \int_a^x G(t)\,dt$ where G has bounded variation on the interval $[a,b]$ then*

$$(29) \qquad\qquad \Delta_s^2 F^*\left((a,b)\right) \leq \mathrm{Var}(G,a,b).$$

PROOF. Suppose first that G is nondecreasing on $[a,b]$. Then by the theorem

$$\Delta_s^2 F^*\left((a,b)\right) = \Lambda_F(a,b) \leq G(b) - G(a).$$

In general if $G = G_1 - G_2$ where both G_1 and G_2 are nondecreasing on $[a,b]$ then $F = F_1 - F_2$ where $F_1 = \int G_1$ and $F_2 = \int G_2$. By Lemma 10 then

$$V_s^2(\Delta_s^2 F, (a,b)) \leq V_s^2(\Delta_s^2 F_1, (a,b)) + V_s^2(\Delta_s^2 F_2, (a,b)) \leq \mathrm{Var}(G,a,b)$$

and (29) is proved.

THEOREM 31 *Let f be a measurable function and E a measurable set. Then if we write $\xi(x,h) = f(x)h^2$ for all $x \in E$ and $h > 0$ we have*

$$\xi^*(E) = \int_E |f(t)|\,dt.$$

PROOF. We may suppose that E is bounded and $\overline{E} \subset (a,b)$. Write $g(x) = |f(x)|\chi_E(x)$, $G_1(x) = \int_a^x g(t)\,dt$ and $G(x) = \int_a^x G_1(t)\,dt$. By Theorem 26 we have $\Delta_s^2 G \equiv g(\Delta\ell)^2$ in (a,b) so that in particular $\Delta_s^2 G \equiv f(\Delta\ell)^2$ in E and $\Delta_s^2 G \equiv 0$ in $(a,b) \setminus E$. Thus $\Delta_s^2 G^*\left((a,b) \setminus E\right) = 0$ and so, using Theorem 29,

$$\Delta_s^2 G^*(E) = \Delta_s^2 G^*\left((a,b)\right) = \Lambda_G(a,b) = \int_E |f(t)|\,dt.$$

The final assertion follows since $\Delta_s^2 G \equiv \xi$ in E.

COROLLARY 32 *If $D_2 F(x) = f(x)$ everywhere in a measurable set E then*

$$\Delta_s^2 F^*(E) = \int_E |f(t)| \, dt.$$

PROOF. This is immediate from the theorem and Theorems 23 and 14.

THEOREM 33 *Suppose that*

$$-k < \underline{D}_2 F(x) \leq \overline{D}_2 F(x) < k$$

at every point of a set E. Then $\Delta_s^2 F^*(E) \leq k|E|$.

PROOF. We may suppose that E is bounded, say that $\overline{E} \subset (a, b)$. Let $\epsilon > 0$ and choose an open set $O \supset E$ with $|O| < |E| + \epsilon$. Set

$$g(x) = \int_a^x c\chi_O(t) \, dt$$

where $c > k$ and

$$G(x) = \int_a^x g(t) \, dt.$$

Note that $D_2 G(x) = c$ at each point x in O, that G is convex and that $\Lambda_G(a, b) \leq c|O|$.

Since by hypothesis $|\Delta_s^2 F(x, h)| < kh^2$ for every $x \in E$ and every suffi­ciently small h we see (from the fact that $D_2 G(x) = c > k$) that

$$\left| \Delta_s^2 F(x, h) \right| < \Delta_s^2 G(x, h)$$

for every $x \in E$ and small h. Thus

$$\Delta_s^2 F^*(E) \leq \Lambda_G(a, b) \leq c|O| < c(|E| + \epsilon).$$

Now, letting $c \to k$ and $\epsilon \to 0+$ completes the proof.

COROLLARY 34 *Suppose that*

$$-\infty < \underline{D}_2 F(x) \leq \overline{D}_2 F(x) < +\infty$$

at every point of a set E. Then $\Delta_s^2 F^*$ is σ–finite on E and vanishes on every subset of E of Lebesgue measure zero.

PROOF. This follows directly from the theorem since the set E may be written as the union of the sets E_n $(n = 1, 2, 3, \ldots)$ where

$$E_n = \left\{ x \in E; \; -n < \underline{D}_2 \, F(x) \le \overline{D}_2 \, F(x) < n \right\}.$$

The estimate in Theorem 33 also goes partially in the opposite direction.

THEOREM 35 *Suppose that* $\overline{D}_2 \, F(x) > k$ *or that* $\underline{D}_2 \, F(x) < -k$ *at every point of a set E. Then* $2\Delta_s^2 F^*(E) \ge k|E|$.

PROOF. We may suppose that E is bounded, say that $\overline{E} \subset (a, b)$. Let $\epsilon > 0$. Let G be any function convex on an open interval containing $[a, b]$ and such that

$$\left| \Delta_s^2 F(x, h) \right| < \Delta_s^2 G(x, h)$$

if $x \in E$ and $0 < h < \delta(x)$.

The collection of intervals of the form $[x - h, x + h] \subset (a, b)$ with $x \in E$, $0 < h < \delta(x)$ and

$$\left| \Delta_s^2 F(x, h) \right| > kh^2$$

is, by the hypotheses of the theorem, a Vitali cover of E. Select a finite disjoint sequence of such intervals $\{[x_i - h_i, x_i + h_i]\}$ $(i = 1, 2, \ldots, n)$ so that

$$|E| - \epsilon < 2 \sum h_i.$$

Then we must have

$$|E| - \epsilon < 2 \sum h_i \le \sum_{i=1}^{n} 2k^{-1} \left| \Delta_s^2 F(x_i, h_i) \right| / h_i$$

$$< 2k^{-1} \sum_{i=1}^{n} \left| \Delta_s^2 G(x_i, h_i) \right| / h_i$$

$$< 2k^{-1} \sum_{i=1}^{n} G'_-(x_i + h_i) - G'_+(x_i - h_i))$$

$$< 2k^{-1} \Lambda_G(a, b).$$

As this holds for all such functions G it follows that

$$|E| - \epsilon \le 2k^{-1} \Delta_s^2 F^*(E)$$

and hence, letting $\epsilon \to 0+$, we obtain $2\Delta_s^2 F^*(E) \ge k|E|$ as required.

22

COROLLARY 36 *Let F be continuous and suppose that the measure $\Delta_s^2 F^*$ is σ–finite on a set E. Then for almost every point $x \in E$*

$$\Delta_s^2 F(x, h) = O(h^2)$$

as $h \to 0+$.

PROOF. We may assume that $\Delta_s^2 F^*(E) < +\infty$. Let us show that for a.e. point $x \in E$ the upper derivate $\overline{D}_2 F(x) < +\infty$. Let E_0 denote the set of points $x \in E$ at which $\overline{D}_2 F(x) = +\infty$. Then by the theorem

$$k|E_0| \le 2\Delta_s^2 F^*(E_0) < +\infty$$

for every $k > 0$. It follows that $|E_0| = 0$ and so $\overline{D}_2 F(x) < +\infty$ for a.e. point $x \in E$. In a similar way it may be shown that $\underline{D}_2 F(x) > -\infty$ for a.e. point $x \in E$. The corollary now follows.

COROLLARY 37 *Let F be continuous and suppose that the measure $\Delta_s^2 F^*$ is σ–finite on a set E. Then both the derivative $F'(x)$ and the second order symmetric derivative $D_2 F(x)$ exist at almost every point of E, and moreover $AD F'(x) = D_2 F(x)$ a.e. in E.*

PROOF. Here $AD F'(x)$ denotes the approximate derivative of the function F' which is defined almost everywhere in E. This follows from the preceding corollary by using a theorem of Marcinkiewicz and Zygmund (see [46, Vol. I, pp. 78–80]).

The next two theorems can be considered as generalized versions of Theorem 23.

THEOREM 38 *Let F be continuous. If F is smooth in a set E, if*

$$-\infty < \underline{D}_2 F(x) \le \overline{D}_2 F(x) < +\infty$$

nearly everywhere in E and $D_2 F(x) = f(x)$ almost everywhere in E then $\Delta_s^2 F \equiv f(\Delta \ell)^2$ in E.

PROOF. Write $\xi_1 = \Delta_s^2 F$ and $\xi_2 = f(\Delta \ell)^2$. Let E_1 be the set of points x in E at which $D_2 F(x) = f(x)$; let E_2 be the set of points x in $E \setminus E_1$ at which the derivates $\underline{D}_2 F(x)$ and $\overline{D}_2 F(x)$ are finite, and let $E_3 = E \setminus (E_1 \cup E_2)$.

By Theorem 23 $(\xi_1 - \xi_2)^*(E_1) = 0$. By Corollary 34 and Theorem 24 $\xi_1^*(E_2) = \xi_2^*(E_2) = 0$ since $|E_2| = 0$ by hypothesis. Finally, by Theorem 28, $\xi_1^*(E_3) = \xi_2^*(E_3) = 0$ since E_3 is countable and F is smooth (Theorem 19). Putting these together we easily obtain $(\xi_1 - \xi_2)^*(E) = 0$ which translates back to the statement

$$V_s^2(\Delta_s^2 F - f(\Delta \ell)^2, E) = 0$$

and this is the conclusion of the theorem.

THEOREM 39 *The relation $\Delta_s^2 F \equiv f(\Delta \ell)^2$ in E holds if and only if the identity $D_2 F(x) = f(x)$ is true both almost everywhere and $\Delta_s^2 F^*$–almost everywhere in E.*

PROOF. If $D_2 F(x) = f(x)$ is true for all $x \in E \setminus N$ then $\Delta_s^2 F \equiv f(\Delta \ell)^2$ in $E \setminus N$. If in addition $\Delta_s^2 F^*(N) = |N| = 0$ then certainly $\Delta_s^2 F \equiv f(\Delta \ell)^2$ in E. Conversely if $\Delta_s^2 F \equiv f(\Delta \ell)^2$ in E then there is, by Theorem 19, a set N of Lebesgue measure zero so that $D_2 F(x) = f(x)$ $(x \in E \setminus N)$. Since N has measure zero and $\Delta_s^2 F^*(N) = (f(\Delta \ell)^2)^*(N)$ it follows from Theorem 24 that $\Delta_s^2 F^*(N) = 0$ as required.

7 The integral

The integral we define in this section is motivated by the problem of determining (up to a linear function) a continuous function F from a function f if $D_2 F(x) = f(x)$ everywhere in an interval $[a, b]$. Interpreted in this way we should produce a second order integral. This is the approach taken in the James integral. A first order integral version is available if the problem is adjusted to require rather the determination of $\Lambda_F(a, b)$. This is the approach of Mařík [35] and we shall follow his development although in a variational setting rather than a Perron setting.

40 DEFINITION. A finite function defined everywhere on an interval (a, b) is said to be (V_s^2)–integrable on $[a, b]$ if there is a continuous function F on $[a, b]$ so that $\Delta_s^2 F \equiv f(\Delta \ell)^2$ in (a, b) and so that

$$(30) \qquad \Lambda_F(a, b) = \lim_{h \to 0+} \frac{F(a) - F(a + h) - F(b - h) + F(b))}{h}$$

exists. We write then

$$\Lambda_F(a, b) = (\text{V}_s^2) \int_a^b f(x)\, dx.$$

The definition requires a justification that (30) does not depend on the choice of function F; this is supplied by Corollary 18. The prefix "(V_s^2)" can frequently be omitted since, as we shall see, the integral is compatible with the common integrals on the real line. The function F is called a *second primitive* for f. If F is a second primitive then we shall see that F' exists almost everywhere and can usually be considered some kind of a first order primitive for f.

Here are a number of elementary properties of the integral. The proofs are almost immediate consequences of the theorems of Section 6; we can omit most of the details.

THEOREM 41 *Let f and g be (V_s^2)–integrable on an interval $[a, b]$. Then so too is any linear combination $sf + tg$ and*

$$\int_a^b \left(sf(x) + tg(x) \right)\, dx = s \int_a^b f(x)\, dx + t \int_a^b g(x)\, dx$$

in the sense of this integral.

PROOF. This follows from the linearity of the equivalence relation and the linearity expressed in (10).

THEOREM 42 *Let f be integrable in either the Riemann, Lebesgue or Perron senses on an interval $[a, b]$. Then f has an (V_s^2)– integral on that interval and*

$$\int_a^b f(x)\, dx$$

has the same value in any of these senses.

PROOF. This follows from Theorem 26.

THEOREM 43 *Suppose that f is (V_s^2)–integrable on $[a, b]$ and nonnegative. Then f is Lebesgue integrable there.*

PROOF. This follows from Theorem 27

THEOREM 44 *Let f be (V_s^2)–integrable on an interval $[a, b]$. Then f is measurable.*

PROOF. This follows from Corollary 20.

THEOREM 45 *Let f be (V_s^2)–integrable on an interval $[a, b]$ and suppose that $f = g$ almost everywhere in $[a, b]$ then g is (V_s^2)–integrable on $[a, b]$ and*

$$\int_a^b f(x)\, dx = \int_a^b g(x)\, dx.$$

PROOF. This follows from Corollary 25.

THEOREM 46 *Let f be (V_s^2)–integrable on an interval $[a, b]$. Then there is a set B of full measure in (a, b) and f is (V_s^2)–integrable on $[c, d]$ for all c, $d \in B$.*

PROOF. If F is a second primitive of f on $[a, b]$ then, by Corollary 37, F' exists almost everywhere in that interval, say on a set B of full measure in (a, b). Using equation (11) we see that $\Lambda_F(c, d) = F'(d) - F'(c)$ must exist for c, $d \in B$. It now follows that f is is (V_s^2)–integrable on $[c, d]$ for all such c, d as required.

In regards to this theorem it is well to point out that there is no claim here that the integral $\int_a^x f(t)\, dt$ exists for almost every $x \in (a, b)$ as is the case with some such integrals. In Section 8 an example is given to illustrate this.

The next two theorems are the major differentiation results for this integral. This first shows that the integrand is the derivative of the second primitive and the second shows how an exact second symmetric derivative may be integrated.

THEOREM 47 *Let f be (V_s^2)–integrable on $[a, b]$ and suppose that F is a second primitive for f. Then F is continuous on $[a, b]$, a.e. differentiable and smooth in (a, b), and $D_2 F(x) = f(x)$ almost everywhere in (a, b). Moreover F' is almost everywhere approximately differentiable and $AD\, F'(x) = f(x)$ almost everywhere in (a, b)..*

PROOF. This follows from Corollary 37.

THEOREM 48 *Let F be continuous on $[a, b]$. Suppose that F is smooth in (a, b),*

$$-\infty < \underline{D_2}\, F(x) \leq \overline{D}_2\, F(x) < +\infty$$

nearly everywhere and that $\Lambda_F(a, b)$ exists. Then $D_2\, F(x) = f(x)$ exists almost everywhere in (a, b), f is (V_s^2)–integrable on $[a, b]$ and

$$(V_s^2) \int_a^b f(t)\, dt = \Lambda_F(a, b).$$

PROOF. This follows from Theorem 38. Note that here f is defined only almost everywhere and so we must either assign an arbitrary value to $f(x)$ where this derivative does not exist or extend (using Theorem 45) the integral to functions defined almost everywhere.

The final property of the (V_s^2)–integral will be useful in applications to trigonometric series. Note that the expression $F(2\pi) + F(-2\pi) - 2F(0)$ that appears in (31) may also be written as $\Delta_s^2 F(0, 2\pi)$.

THEOREM 49 *Suppose that f is 2π–periodic and that there is a continuous function F on the interval $[-2\pi, 2\pi]$ so that $\Delta_s^2 F \equiv f(\Delta\ell)^2$ in $(-2\pi, 2\pi)$. Then*

$$(31) \qquad \Lambda_F(0, 2\pi) = -\frac{F(2\pi) + F(-2\pi) - 2F(0)}{2\pi},$$

f is (V_s^2)–integrable on $[0, 2\pi]$ and

$$(V_s^2) \int_0^{2\pi} f(t)\, dt = -\frac{F(2\pi) + F(-2\pi) - 2F(0)}{2\pi}.$$

PROOF. It is enough to establish (31) since $\Delta_s^2 F \equiv f(\Delta\ell)^2$ in $(0, 2\pi)$ as well. We can simplify our computations by assuming that F is normalized on $[-2\pi, 2\pi]$ by requiring that $F(-2\pi) = F(2\pi) = 0$. Write $G(x) = F(x + 2\pi)$; then, since f is 2π–periodic, we have $\Delta_s^2 G \equiv f(\Delta\ell)^2$ in $(-2\pi, 0)$ so that $\Delta_s^2 G \equiv \Delta_s^2 F$ in $(-2\pi, 0)$. Consequently $F(x + 2\pi) - F(x)$ is linear on that interval. From this we obtain that

$$(32) \qquad F(x + 2\pi) = F(x) - F(0) - \frac{F(0)}{\pi} x.$$

Now then directly from the definition of $\Lambda_F(0, 2\pi)$, from (32) and from the fact that F must be smooth at 0 we obtain

$$\Lambda_F(0, 2\pi) = \lim_{h \to 0+} -\frac{F(2\pi - h) + F(h) - F(2\pi) - F(0)}{h}$$

$$= \lim_{h \to 0+} -\frac{F(h) + F(-h) - F(0)}{h} - \frac{F(0)}{\pi} = -\frac{F(0)}{\pi}$$

which is the required conclusion (as $F(-2\pi) = F(2\pi) = 0$). The final integration statement follows immediately from this since F is a continuous second primitive of f on $[0, 2\pi]$.

8   Additivity

The integral does not quite have the usual additivity properties one expects to encounter in integration theories. In part this comes from the symmetric nature of the integral and in part from the fact that conditions ensuring integrability on two abutting intervals may break down at the common point.

A simple example (cf. [35, p. 58]) illustrates one aspect of the problem. Let

(33) $$F(x) = x\sqrt{1 - x^2} \quad (-1 \le x \le 1).$$

Then while the integral $(V_s^2) \int_{-1}^1 F''(t)\, dt$ exists this function is not integrable on $[-1, c]$ for any value of $c \in (-1, 1)$. Note that the integral does exist on $[c, d]$ for any $-1 < c < d < 1$. This example illustrates how the (V_s^2)–integral exploits the symmetry at the points -1 and 1.

It is also possible for the integral to exist on $[a, b]$ and on $[b, c]$ but not exist on the interval $[a, c]$. Extend the function F in example (33) so as to be zero everywhere outside of $[-1, 1]$. Then $f = F''$ is (V_s^2)–integrable on each of the intervals $[-1, 1]$ and $[1, 2]$ but not integrable on $[-1, 2]$. It is the lack of smoothness of F at 1 that produces the problem.

If we assume integrability then additivity is easy enough to see as the first theorem shows.

THEOREM 50 *If f is (V_s^2)–integrable on $[a, b]$ and on $[a, c]$ $(a < b < c)$ then it is integrable on $[b, c]$ and*

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx$$

in the sense of this integral.

PROOF. Let F be a second primitive for f on $[a, c]$. By hypothesis $\Lambda_F(a, b)$ and $\Lambda_F(a, c)$ exist and, by Corollary 37, F is smooth. This is enough to establish that

(34) $$\Lambda_F(b, c) = \Lambda_F(a, c) - \Lambda_F(a, b)$$

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and the theorem is proved. Check the following identity

$$F(b) - F(b + h) - F(c - h) + F(c)$$

$$= (F(a) - F(a + h) - F(c - h) + F(c))$$

$$- (F(a) - F(a + h) - F(b - h) + F(b)) + \Delta_s^2 F(b, h)$$

and then (34) follows readily.

The example (33) above, illustrating that a function may be integrable on two abutting intervals $[a, b]$ and $[b, c]$ and yet not on the interval $[a, c]$, also contains the clue to when this is possible. If the conditions exist for a "smooth" join then the additivity is available.

THEOREM 51 *Let $a < c < b$ and suppose that f is (V_s^2)–integrable on $[a, c]$ with a second primitive F and on $[c, b]$ with a second primitive G. Then f is (V_s^2)–integrable on $[a, b]$ if and only if the limit*

$$(35) \qquad \lim_{h \to 0+} \frac{F(c) - F(c - h) - G(c + h) + G(c)}{h}$$

exists.

PROOF. If the function f is (V_s^2)–integrable on $[a, b]$ then there is a continuous H with $\Delta_s^2 H \equiv f(\Delta \ell)^2$ in (a, b). In particular $\Delta_s^2 H \equiv \Delta_s^2 F$ in (a, c) and $\Delta_s^2 H \equiv \Delta_s^2 G$ in (c, b). Thus $H - F$ is linear in the first interval and $H - G$ is linear in the second. Write $H(x) = F(x) + \alpha_1 x + \beta_1$ $(a \leq x \leq c)$ and $H(x) = G(x) + \alpha_2 x + \beta_2$ $(c \leq x \leq b)$. The condition that H is smooth at c now translates directly into the condition that the limit in (35) is $\alpha_2 - \alpha_1$.

Conversely suppose that the limit in (35) exists and is α. We write $H(x) = F(x)$ $(a \leq x \leq c)$ and

$$H(x) = F(c) + G(x) - G(c) + \alpha(x - c) \quad (a \leq x \leq c).$$

Then $\Delta_s^2 H \equiv f(\Delta \ell)^2$ in $(a, b) \setminus \{c\}$. It is straightforward now to check, using the fact that the limit in (35) is α, that H is smooth at c. This gives the relation $\Delta_s^2 H \equiv f(\Delta \ell)^2$ in (a, b).

For f to be integrable we need now only the existence of $\Lambda_H(a, b)$. But, using (13) and the fact that H is smooth at c, we have

$$\Lambda_F(a, c) + \Lambda_G(c, b) = \Lambda_H(a, c) + \Lambda_H(c, b) = \Lambda_H(a, b)$$

and so the theorem is proved.

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9 Relations to the James P^2–integral

In this section we shall present the classical definition of the P^2–integral as given in Zygmund [46, Vol. II, pp. 86–91] and show how it relates to the V_s^2–integral. There are a number of equivalent definitions in the literature. It might be noted that the version cited in Skljarenko [38] is more restrictive since he assumes that the major and minor functions are smooth.

Let f be a finite valued function defined everywhere on an interval (a, b). A function M is a *second symmetric major function* for f if M is continuous on $[a, b]$, if $M(a) = M(b) = 0$ and if $\underline{D}_2\, M(x) \geq f(x)$ for each $a < x < b$. A function N is a *second symmetric minor function* for f if $-M$ is a second symmetric major function for $-f$. It can be seen that $M(x) \leq N(x)$ for any such pair. If, for some value of $c \in (a, b)$,

$$\sup_M M(c) = \inf_N N(c)$$

then it can be shown that $F(x) = \sup_M M(x) = \inf_N N(x)$ exists for all $a \leq x \leq b$. In this case f is said to be P^2–integrable on $[a, b]$ and the function F is called its second indefinite P^2–integral on $[a, b]$. In the usual notation one writes

$$F(c) = (\mathrm{P}^2) \int_{a,b,c} f(t)\, dt \qquad (a \leq c \leq b).$$

The original notation in [24] and [26] differs; we follow Zygmund here. For further details and a development of this integral directly from the definition see this work. We wish to relate these concerns directly to the earlier material on symmetric variation. The first lemma is a step towards this; note that this lemma does not quite assert that $\Delta_s^2 F \equiv f(\Delta\ell)^2$ on (a, b) which is our goal.

LEMMA 52 *A finite function defined everywhere on an interval (a, b) is P^2–integrable on $[a, b]$ if and only if there is a continuous function F on $[a, b]$ with the following property: for every $\epsilon > 0$ there is a function G convex and continuous on $[a, b]$ with $G(a) = G(b) = 0$, with $-\epsilon < G(x)$ $(a < x < b)$ and such that*

$$\left| \Delta_s^2 F(x, h) - f(x)h^2 \right| < \Delta_s^2 G(x, h) \quad (a < x < b)$$

for all sufficiently small $h > 0$. In this case

$$H(x) = F(x) - F(a) - (F(b) - F(a))(x - a)/(b - a)$$

is a P^2–indefinite integral for f on $[a,b]$.

PROOF. A proof can be found in Skljarenko [38]. It is straightforward in any case.

THEOREM 53 *A finite function defined everywhere on an interval (a,b) is P^2–integrable on $[a,b]$ if and only if there is a continuous function F on $[a,b]$ so that $\Delta_s^2 F \equiv f(\Delta\ell)^2$ on (a,b). In this case*

$$H(x) = F(x) - F(a) - (F(b) - F(a))(x-a)/(b-a)$$

is a P^2–indefinite integral for f on $[a,b]$.

PROOF. The sufficiency of this condition is evident in view of Lemma 52. To prove that the condition is necessary let us suppose that f is P^2–integrable on $[a,b]$. Let $\epsilon > 0$ and fix $a < c < d < b$. By Lemma 52 there is a continuous, convex G on $[a,b]$ with $-\epsilon < G(x)$ $(a < x < b)$, with $G(a) = G(b) = 0$ and so that

$$\left| \Delta_s^2 F(x,h) - f(x)h^2 \right| < \Delta_s^2 G(x,h) \quad (a < x < b)$$

for all sufficiently small $h > 0$. Note that $G'_+(c) > -\epsilon(c-a)^{-1}$ and $G'_-(d) < \epsilon(b-d)^{-1}$ and consequently

$$\Lambda_G(c,d) = G'_-(d) - G'_+(c) < \epsilon(c-a)^{-1} + \epsilon(b-d)^{-1}.$$

Comparing with Definition 6 and letting $\epsilon \to 0+$ we see that we have proved that

$$\mathrm{V}_s^2(\Delta_s^2 F - f(\Delta\ell)^2, (c,d)) = 0.$$

Now the interval (a,b) may be expressed as the union of the intervals $(a + n^{-1}, b - n^{-1})$ for $n = 1,2,3\ldots$ and $\Delta_s^2 F \equiv f(\Delta\ell)^2$ on each such interval. By Lemma 11 it follows that $\Delta_s^2 F \equiv f(\Delta\ell)^2$ on (a,b) as required.

Theorem 53 shows that there is an intimate relation between the P^2–integral and the (V_s^2)–integral. From Theorem 53 we have trivially that a function that is (V_s^2)–integrable must be also P^2–integrable. This can go occasionally in the other direction too. We express these observations as corollaries.

COROLLARY 54 *A function f defined everywhere on an interval (a,b) is V_s^2–integrable on $[a,b]$ if and only if it is P^2–integrable on $[a,b]$ and $\Lambda_F(a,b)$ exists where F is a P^2–indefinite integral for f on $[a,b]$.*

PROOF. This follows immediately from the theorem.

COROLLARY 55 *Suppose that a function f is P^2–integrable on an interval $[a, b]$ with a P^2–primitive F. Then $F'(x)$ exists for every x in a set B of full measure in (a, b), f is V_s^2–integrable in $[c, d]$ for all $c, d \in B$ and*

$$(36) \qquad (\mathrm{V}_s^2) \int_c^d f(t)\, dt = F'(d) - F'(c)$$

for all $c, d \in B$.

PROOF. This follows from the theorem together with Corollary 37 and equation (11).

COROLLARY 56 *Suppose that f is 2π–periodic and P^2–integrable on the interval $[-2\pi, 2\pi]$ with P^2–primitive F. Then f is (V_s^2)–integrable on $[0, 2\pi]$ and*

$$(\mathrm{V}_s^2) \int_0^{2\pi} f(t)\, dt = -\frac{F(2\pi) + F(-2\pi) - 2F(0)}{2\pi}.$$

so that, in particular,

$$(\mathrm{V}_s^2) \int_0^{2\pi} f(t)\, dt = -\frac{1}{\pi} \int_{-2\pi, 2\pi, 0} f(t)\, dt.$$

PROOF. This follows from the theorem together with Theorem 49.

10   Relations to the Burkill (SCP)–integral

In this section we present an account of the (SCP)–integral of Burkill and show its relation with the integral of the preceding section. In Burkill's original account he makes a smoothness assumption on the major and minor functions which obscures the connection with the P^2–integral. The version presented here is formally more general and would include any of the other variants which might be found in the literature. For the role of smoothness conditions in this and the higher order James integrals see [20].

The (SCP)–integral is based on the symmetric Cesàro derivative which is really just a version of the second order symmetric derivative. If a function G is integrable in any appropriate sense (usually taken as the Denjoy-Perron

integral) with an indefinite integral F then one writes for the symmetric Cesàro derivates

$$\underline{\text{SCD}}\, G(x) = \underline{\text{D}}_2\, F(x) \quad \text{and} \quad \overline{\text{SCD}}\, G(x) = \overline{\text{D}}_2\, F(x).$$

We say that G is (SC)–*lower semicontinuous* at a point x if

$$\liminf_{h \to 0+} \Delta_s^2 F(x, h)/h \geq 0$$

and that G is (SC)–*upper semicontinuous* at x if

$$\limsup_{h \to 0+} \Delta_s^2 F(x, h)/h \leq 0.$$

If both conditions hold G is said to be (SC)–*continuous* which is evidently just the statement that F is smooth at x. We say that G is (C)–*continuous* at a point if G is the derivative of F at that point. (C)–continuity at the endpoints of an interval is normally interpreted in a one-sided sense.

Write, for any integrable function G with $F = \int G$,

$$\Theta_s^2 G(x, h) = \Delta_s^2 F(x, h).$$

Explicitly then

$$\Theta_s^2 G(x, h) = \int_x^{x+h} G(t)\, dt - \int_{x-h}^x G(t)\, dt = \int_0^h \left(G(x+t) - G(x-t) \right)\, dt.$$

Now we see that G is (SC)–continuous at a point x if $\Theta_s^2 G(x, h) = o(h)$ as $h \to 0+$; it is (SC)–lower semicontinuous or (SC)–upper semicontinuous accordingly as

$$\liminf_{h \to 0+} \Theta_s^2 G(x, h)/h \geq 0$$

or

$$\limsup_{h \to 0+} \Theta_s^2 G(x, h)/h \leq 0.$$

The SCD–derivative is the limit of the ratio $\Theta_s^2 G(x, h)/h^2$. These notions, then, are defined as direct properties of G without an explicit reference to the primitive function F.

A standard Perron type integral based on this derivative is defined in Burkill [11] and termed the (SCP)–integral. A brief version is now given here. Note that because of the relaxed conditions on the (SC)–continuity

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(assumption (2.) below) this definition is formally more general than the original one in [11].

Let f be a finite valued function in an interval (a, b) and B a set of full measure in $[a, b]$ containing both points a and b. A function M is said to be an (SCP)–major function for f on $[a, b]$ with basis B if

1. $M(a) = 0$.

2. M is (SC)–lower semicontinuous everywhere in (a, b).

3. M is (C)–continuous in B.

4. $\underline{\text{SCD}}\, M(x) \geq f(x)$ for almost every $a < x < b$.

5. $\underline{\text{SCD}}\, M(x) > -\infty$ for nearly every $a < x < b$ (i.e. except possibly in a countable set N).

Note that the conditions (4) and (5) already show that M must be (SC)–lower semicontinuous except possibly at the points where SCD $M(x) = -\infty$; the force of assumption (2) is to get semicontinuity at these points too. It is here that our definition differs from the original of Burkill; he assumes instead (SC)–continuity at each point of (a, b) and this places an extra superfluous assumption on the major and minor functions.

A function m is said to be an (SCP)–minor function for f on $[a, b]$ with basis B if $-m$ is an (SCP)–major function for $-f$ on $[a, b]$ with basis B. The usual Perron-type definitions now produce an integral. If

$$I = \inf M(b) = \sup m(b)$$

where the infimum is taken over all (SCP)–major functions M for f and the supremum is taken over all (SCP)–minor functions m for f then f is said to be (SCP)–*integrable* on $[a, b]$ with basis B and we write

$$I = (\text{SCP,}B) \int_a^b f(x)\, dx.$$

Reference to the basis may be suppressed with the understanding that some set of full measure in $[a, b]$ is to be employed.

The justification for this integral requires an appeal to a monotonicity theorem. This is supplied by the following lemma.

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LEMMA 57 *Let F be continuous function on an interval $[a, b]$, suppose that $\underline{D}_2 F(x) \geq 0$ a.e. in (a, b), $\underline{D}_2 F(x) > -\infty$ except on a countable set N and*

$$\liminf_{h \to 0+} \Delta_s^2 F(x, h)/h \geq 0$$

at each point of N. Then F is convex.

PROOF. This can be proved as in [1, Vol. II, p. 344] or [46, Vol. I, p. 328].

Applied to the difference $M - m$ of a pair of major and minor functions in this sense this lemma shows that $\int (M - m)$ is convex and hence that $M - m$ is equivalent to a nondecreasing function and so, in particular, nondecreasing on the basis B.

It can be shown (see [11]) that if f is (SCP,B)–integrable on $[a, b]$ then f is (SCP,B)–integrable on $[a, x]$ for all $a < x \leq b$, $x \in B$. The function

$$G(x) = (\text{SCP},B) \int_a^x f(t)\,dt$$

is called the (SCP)–indefinite integral of f and is defined almost everywhere.

Let us begin our study by dispensing immediately with the exceptional sets. Normally in Perron theories these exceptional sets are irritating to deal with and can be removed from the theory with no loss of generality. For the purposes of this lemma only we shall call a function M a *strong* (SCP)–major function for f on $[a, b]$ with basis B if

1. $M(a) = 0$.

2. M is (C)–continuous in B.

3. $\underline{\text{SCD}}\, M(x) \geq f(x)$ for every $a < x < b$.

A strong (SCP)–minor function is similarly defined. Note that a strong (SCP)–major function is certainly a (SCP)–major function in the ordinary sense. The integral that results from using this definition can be called a strong (SCP)–integral. As a result of the lemma we now prove we may always take (SCP)–major/minor functions as strong.

LEMMA 58 *The strong (SCP)–integral and the ordinary (SCP)–integral are equivalent.*

PROOF. It is clear that the ordinary (SCP)–integral includes the strong version. Suppose that f is (SCP)–integrable (ordinary sense) on $[a,b]$. Let $\epsilon > 0$. Take any major/minor functions M and m in this sense with $M(b) - m(b) < \epsilon/3$. Let C be the set of points $x \in (a,b)$ where either $\underline{\mathrm{SCD}}\,M(x) = -\infty$ or $\overline{\mathrm{SCD}}\,m(x) = +\infty$. By definition C is countable. Let E be the set of points $x \in (a,b)$ where either $f(x) > \underline{\mathrm{SCD}}\,M(x) > -\infty$ or $f(x) < \overline{\mathrm{SCD}}\,m(x) < +\infty$. By definition $|E| = 0$.

We use Lemma 9 applied to the countable set C to obtain a convex function K on an interval containing $[a,b]$ so that $\Lambda_K(a,b) < \epsilon/6$ and so that

$$\Delta_s^2 K(x,h) \geq f(x)h^2 - \Theta_s^2 M(x,h)$$

and

$$\Delta_s^2 K(x,h) \geq \Theta_s^2 m(x,h) - f(x)h^2$$

for $x \in C$ and sufficiently small h. Since C is countable and M and m are (SC)–semicontinuous at each point of C this allows us to apply Lemma 9. Now define a monotonic function K_1 so that $K_1(a) = 0$, $K_1(b) < \epsilon/6$ and $K(x) = \int_a^x K_1(t)\,dt$. We can insist that the function K_1 is (C)–continuous at a and b by subtracting appropriate linear functions from K and choosing the values $K_1(a)$ and $K_1(b)$ correctly.

Similarly we use Lemma 7 applied to the measure zero set E to obtain a convex function H on an interval containing $[a,b]$ so that $\Lambda_H(a,b) < \epsilon/6$ and so that

$$\Delta_s^2 H(x,h) \geq f(x)h^2 - \Theta_s^2 M(x,h)$$

and

$$\Delta_s^2 H(x,h) \geq \Theta_s^2 m(x,h) - f(x)h^2$$

for $x \in E$ and sufficiently small h. Again define a monotonic function H_1 so that $H_1(a) = 0$, $H(x) = \int_a^x H_1(t)\,dt$ and $H_1(b) < \epsilon/6$; we can require H_1 to be (C)–continuous at a and b.

Write $M_1(x) = M(x) + K_1(x) + H_1(x)$ and $m_1(x) = m(x) - K_1(x) - H_1(x)$. It is straightforward now to verify that M_1 and m_1 are a major/minor function pair for f in the strong sense.

Moreover $M_1(b) - m_1(b) = M(b) - m(b) + 2K_1(b) + 2H_1(b) < \epsilon$. From this it is clear that the integrability of f in the ordinary (SCP)–sense requires f to be integrable in the narrower sense too as required.

The following variational characterization of the (SCP)–integral is the key to establishing the relation to the variational theory given here. This follows a standard procedure known for most Perron type integrals.

LEMMA 59 *A finite function f defined everywhere on an interval (a, b) is* (SCP)–*integrable on* $[a, b]$ *with an indefinite integral G if and only if*

1. $G(a) = 0$.

2. G *is* (C)–*continuous a.e. in* (a, b) *and at a and b.*

3. G *is* (SC)–*continuous in* (a, b).

4. G *is Denjoy-Perron integrable in* $[a, b]$.

5. *for every $\epsilon > 0$ there is nonnegative, increasing function L that is* (C)–*continuous at a and b with $L(a) = 0$, $L(b) < \epsilon$ so that*

$$\left| \Theta_s^2 G(x, h) - f(x) h^2 \right| < \Theta_s^2 L(x, h)$$

for each $a < x < b$ and all sufficiently small $h > 0$.

PROOF. The conditions are sufficient. If (1)–(5) hold then using successively $\epsilon = 1, 1/2, 1/3, \ldots 1/n$ there is a sequence of such functions L_n. Write $M_n = G + L_n$ and $m_n = G - L_n$. For the basis take B as the set of points where G is (C)–continuous and all L_n are too.

Clearly $M_n(a) = m_n(a) = 0$, $m_n(x) \leq G(x) \leq M_n(x)$ almost everywhere and $M_n(b) - m_n(b) < 1/n$. Both M_n and m_n are (C)–continuous in B. Condition (5) ensures that

$$\Theta_s^2 G(x, h) - \Theta_s^2 L_n(x, h) < f(x) h^2 < \Theta_s^2 G(x, h) + \Theta_s^2 L_n(x, h)$$

at each x and for sufficiently small $h > 0$; it follows that $\underline{\text{SCD}}\, M_n(x) \geq f(x) \geq \overline{\text{SCD}}\, m_n(x)$ at each x. Thus M_n and m_n form sequences of major/minor function pairs for f and the integrability of f follows with an indefinite integral G defined on B.

The conditions are also necessary. Suppose f is (SCP)–integrable on $[a, b]$ with an indefinite integral G. Conditions (1)–(4) for G are straightforward and well known. For example for (3) the function G is the limit of a sequence of (SC)–lower semicontinuous functions and a sequence of (SC)–upper semicontinuous functions. The integrals can be shown to converge uniformly and so the function G is itself (SC)–continuous in (a, b).

We show now that (5) holds too. Let $\epsilon > 0$. Let M and m be any (SCP)–major and minor functions for f on $[a, b]$ with $M(b) - m(b) < \epsilon/2$. Because of Lemma 58 we may take these as strong major/minor functions.

We note that the functions $M - G$ and $G - m$ are nondecreasing (i.e. they are a.e. equivalent to nondecreasing functions). In particular then

$$\Theta_s^2 M(x, h) \geq \Theta_s^2 G(x, h) \geq \Theta_s^2 m(x, h).$$

Write

$$L(x) = M(x) - m(x) + c(x - a)$$

where $2c(b-a) = \epsilon$. Clearly L is equivalent to a monotonic function, $L(a) = 0$ and $L(b) = M(b) - m(b) + c(b-a) < \epsilon$. We can extend L so as to be monotonic on $[a, b]$ and it will be (C)–continuous at a and b.

We show that L satisfies the rest of the statement in (5). Suppose that $x \in (a, b)$. Then, since M and m are (strong) major/minor functions for f, $\underline{\text{SCD}}\, M(x) \geq f(x) \geq \overline{\text{SCD}}\, m(x)$; it follows that, for small enough h,

$$\Theta_s^2 m(x, h) - ch^2 < f(x)h^2 < \Theta_s^2 M(x, h) + ch^2$$

and so

$$|f(x)h^2 - \Theta_s^2 G(x, h)| < \Theta_s^2 L(x, h).$$

exactly as required to verify (5). This completes the proof.

This lemma translates immediately into the language of the previous sections and places the (SCP)–integral in that context.

THEOREM 60 *A function f defined everywhere on an interval (a, b) is (SCP)–integrable on $[a, b]$ if and only if there is a continuous, ACG_* function F on $[a, b]$ with finite one-sided derivatives $F'_+(a)$ and $F'_-(b)$ at the endpoints of the interval so that $\Delta_s^2 F \equiv f(\Delta\ell)^2$ on (a, b). If so then*

$$(\text{SCP}) \int_a^b f(t)\, dt = F'_-(b) - F'_+(a)$$

and the set of points

$$B = \{a\} \cup \{b\} \cup \{x \in (a, b);\ F'(x)\ \text{exists}\ \}$$

may be taken as the basis.

Trivially we see now that any SCP–integrable function is integrable in both the P^2 and V_s^2 senses. The relation with the (V_s^2)–integral and the P^2–integral is now immediate; we express these statements as corollaries. The basis plays no role and need not be explicitly mentioned. Corollary 62 is just Theorem I of [14].

COROLLARY 61 *Suppose that a function f is (SCP)–integrable on an interval $[a,b]$. Then f is (V_s^2)–integrable on $[a,b]$ and*

$$(V_s^2) \int_a^b f(t)\, dt = (SCP) \int_a^b f(t)\, dt.$$

COROLLARY 62 *Suppose that a function f is (SCP)–integrable on an interval $[a,b]$. Then f is P^2–integrable on $[a,b]$. If G is an indefinite (SCP)–integral for f then*

$$F(x) = \int_a^x G(t)\, dt - \frac{x-a}{b-a} \int_a^b G(t)\, dt$$

is a second indefinite integral for f in the P^2–sense on $[a,b]$.

A converse from the P^2 or V_s^2 integrals to the (SCP)–integral just requires that the second order primitive for f satisfy some extra conditions. Viewed in the statement of Theorem 60 the requirement that the primitive be an ACG_* function might be considered entirely arbitrary; the intention is that the primitive be the integral of its derivative. This could well be interpreted in the narrower sense of the Lebesgue integral (in which case F should be absolutely continuous) or in the broader sense of the Denjoy-Khintchine integral (in which case F should be ACG). For applications to trigonometric series the Lebesgue integral would have sufficed; in the setting of these two integrals the Denjoy-Khintchine integral would have been more natural. In this regard see the discussion in Skvorcov [41].

The following examples show the narrowness of the (SCP)–integral. We have used the first example before in Section 8.

Let $F(x) = x\sqrt{1-x^2}$ on the interval $[-1,1]$. Then while the integral

$$(V_s^2) \int_{-1}^1 F''(t)\, dt$$

exists this function cannot be integrable in the (SCP)–sense on $[-1,1]$. While this example is perhaps rather artificial this feature does have some importance. As we shall see in Section 13 a function f that is everywhere the sum of a convergent trigonometric series will be V_s^2–integrable over any period $[a, a+2\pi]$. The best that can be said if one uses the (SCP)–integral is that the function is (SCP)–integrable over a period $[a, a+2\pi]$ for almost every value of a. It is because the V_s^2–integral exploits the symmetry available that the improved statement is possible.

The final example we shall merely cite. Skvorcov [43] has given an example of a continuous function F that has everywhere a second order symmetric derivative $D_2 F(x) = f(x)$ and yet F is not ACG$_*$. Then f is P^2–integrable and also V^2_s–integrable on $[a, b]$ for any a and b at which F' exists (this is almost everywhere) but f is not (SCP)–integrable because F' is not Denjoy-Perron integrable. (In this particular example F is ACG and so F' *is* integrable in the more general Denjoy-Khintchine sense and this leaves open the question as to whether an improved version of the (SCP)–integral using the Denjoy-Khintchine integral in place of the Denjoy-Perron integral would be as far removed from the P^2 and V^2_s–integrals.)

11 Mařík's integration by parts formula

The usual integration by parts formula of the calculus can be considered an interpretation of the differentiation formula $(fg)' = f'g + fg'$ in an integration setting. Most integration by parts formulas, even for generalized integrals, continue the same theme. Mařík's version for the second order symmetric integrals might be viewed instead as an interpretation of the differentiation formula

$$(37) \qquad (GF' - G'F)' = GF'' - G''F.$$

The next three lemmas provide the key to the integration by parts formula. Note that the equivalence relation expressed in (41) is essentially the differentiation relation in (37).

LEMMA 63 *Let*

$$(38) \qquad H(x) = \int_a^x G(t)\, dF(t) - \int_a^x F(t)\, dG(t)$$

where F is continuous and G' is Lipschitz on $[a, b]$. Then, for each $x \in (a, b)$,

$$(39) \qquad \Delta^2_s H(x, h) = G(x)\Delta^2_s F(x, h) - F(x)\Delta^2_s G(x, h) \\ + 2G'(x)M^2_s F(x, h)h^2 + o(h^2)$$

as $h \to 0+$.

PROOF. The proof is from [35, (109), p. 63], reproduced here since the original is unpublished. Fix $x \in (a, b)$ and $h > 0$. We can assume that $F(a) = 0$. An integration by parts establishes the formula

$$H(x) = F(x)G(x) - 2 \int_a^x F(t)G'(t)\, dt$$

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so that
$$\Delta_s^2 H(x, h) =$$

$$\Delta_s^2 FG(x, h) - 2 \int_0^h \left(F(x+t)G'(x+t) - F(x-t)G'(x-t) \right) dt$$

For convenience this expression can be simplified by writing

$$\Omega(f, g) = \Delta_s^2 fg(x, h) - 2 \int_0^h \left(f(x+t)g'(x+t) - f(x-t)g'(x-t) \right) dt$$

which is linear in f and g separately. Set $F(x) = \alpha$, $G(x) = \beta$, $G'(x) = \gamma$, $F_1 = F - \alpha$, $L(t) = t - x$ and $G_1 = G - \beta - \gamma L$. Then

$$\Omega(F, G) = \Omega(F, G_1) + \Omega(F, \beta + \gamma L)$$

and so

(40) $\qquad \Delta_s^2 H(x, h) = \Omega(F_1, G_1) + \alpha \Omega(1, G_1) + \beta \Omega(F, 1) + \gamma \Omega(F, L).$

Now we can check directly that $\Omega(F, L) = 2h^2 M_s^2 F(x, h)$, that $\Omega(1, G_1) = -\Delta_s^2 G(x, h)$ and that $\Omega(F, 1) = \Delta_s^2 F(x, h)$. The remaining term $\Omega(F_1, G_1)$ must be shown to be $o(h^2)$ as $h \to 0+$. Since F_1 is continuous, $F_1(x) = 0$, $G_1(x) = G_1'(x) = 0$ and G'' is bounded we easily show that

$$F_1(x+t)G_1(x+t) = o(t^2)$$

and that

$$\int_0^t F_1(x+s)G_1'(x+s) \, ds = o(t^2)$$

as $t \to 0+$ or $t \to 0-$. Together this shows that $\Omega(F_1, G_1) = o(h^2)$ as $h \to 0+$ as we wished to prove. Now these computations along with (40) provide (39) and the lemma is proved.

LEMMA 64 *Let H be defined as in (38) where F is continuous and G' is Lipschitz on $[a, b]$ and suppose that F has a finite symmetric derivative at every point of a set $E \subset (a, b)$. Then*

(41) $\qquad\qquad\qquad \Delta_s^2 H \equiv G\Delta_s^2 F - F\Delta_s^2 G$

on E.

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PROOF. By Lemma 3, $\mathrm{M}_s^2 F(x,h) = o(1)$ as $h \to 0+$ for each $x \in E$. It follows then that
$$2G'(x)\mathrm{M}_s^2 F(x,h)h^2 = o(h^2)$$
as $h \to 0+$. Now (41) follows immediately from Lemma 63 and Lemma 8.

LEMMA 65 *Let H be defined as in (38) where F is continuous and G' is Lipschitz on $[a,b]$ and suppose that*
$$\limsup_{h \to 0+} |\mathrm{M}_s^2 F(x,h)| < +\infty$$
at every point of a set $E \subset (a,b)$ where E has measure zero. Then (41) holds on E.

PROOF. Write
$$\xi(x,h) = 2G'(x)\mathrm{M}_s^2 F(x,h)h^2.$$
As $\mathrm{M}_s^2 F(x,h) = O(1)$ as $h \to 0+$ for $x \in E$ and E has measure zero it follows from Theorem 24 that $\xi^*(E) = 0$. Now once again (41) follows from Lemma 63 and Lemma 8.

THEOREM 66 (Mařík) *Let f be (V_s^2)–integrable on $[a,b]$ with a second primitive F. Let H be defined as in (38) where G' is Lipschitz on $[a,b]$. Suppose that*
(42) $$\limsup_{h \to 0+} |\mathrm{M}_s^2 F(x,h)| < +\infty$$
at nearly every point of (a,b) and that $\Lambda_H(a,b)$ exists. Then fG is (V_s^2)–integrable on $[a,b]$ and

$$(\mathrm{V}_s^2) \int_a^b f(t)G(t)\,dt = \Lambda_H(a,b) + \int_a^b F(t)G''(t)\,dt$$

where the latter integral is a Lebesgue integral.

PROOF. Let N denote the countable set of points in (a,b) at which (42) fails. We know that F is almost everywhere differentiable in (a,b) and so Lemma 64 and Lemma 65 together show that

$$\Delta_s^2 H \equiv G\Delta_s^2 F - F\Delta_s^2 G$$

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in $(a, b) \setminus N$. In fact we shall show that this equivalence relation holds on (a, b); it is just a matter of handling this countable exceptional set.

By Theorem 47 the function F is smooth and so, since N is countable, $\Delta_s^2 F^*(N) = 0$. It is clear that G is also smooth so that, for the same reason, $\Delta_s^2 G^*(N) = 0$. We show that H is smooth. An integration by parts yields

$$(43) \qquad H(x) = F(x)G(x) - 2 \int_a^x F(t)G'(t)\, dt.$$

The product of a smooth function (F) and a differentiable function (G) is smooth; an indefinite integral with a continuous integrand is smooth. Thus the smoothness of H is clear from (43) and we may conclude that $\Delta_s^2 H^*(N) = 0$. Now $\Delta_s^2 H \equiv G\Delta_s^2 F - F\Delta_s^2 G$ in (a, b) follows.

The relation $\Delta_s^2 F \equiv f(\Delta\ell)^2$ on (a, b) holds because f is integrable and F is its second primitive. As G is bounded it is easy to see that $G\Delta_s^2 F \equiv fG(\Delta\ell)^2$ on (a, b) holds too. Since G'' is integrable $\Delta_s^2 G \equiv G''(\Delta\ell)^2$ and so, since F is bounded too, $F\Delta_s^2 G \equiv FG''(\Delta\ell)^2$ on (a, b) holds. Thus we have $fG(\Delta\ell)^2 \equiv \Delta_s^2 H + FG''(\Delta\ell)^2$ in (a, b). The final assertion of the theorem now follows from this equivalence relation and the existence of $\Lambda_H(a, b)$.

In the statement of this theorem we have required the existence of the expression $\Lambda_H(a, b)$ in order to claim the existence of the integral. Indeed this may not exist so that a product fG may fail to be integrable even if G is linear. This reflects the fragile nature of the integral here in that its existence may arise from a symmetry that a multiplication can destroy. An example (from [35, (119), p. 69]) illustrates: take $F(x)$ as in (33) and $G(x) = x$. Then with H as in (43) it can be shown that $\Lambda_H(-1, 1)$ does not exist. Consequently $F''G$ is not integrable on $[-1, 1]$.

The corollary we now prove gives an instance when $\Lambda_H(a, b)$ must exist and so the hypotheses are simpler. Again this is due to Mařík [35, (116), p. 68]) .

COROLLARY 67 *Let f be (V_s^2)–integrable on $[a, b]$ with a second primitive F. Suppose that G'' is bounded and integrable on $[a, b]$. Suppose that*

$$\limsup_{h \to 0+} |\mathrm{M}_s^2 F(x, h)| < +\infty$$

at nearly every point of (a, b) and that $G(a) = G(b)$. Then fG is (V_s^2)– integrable on $[a, b]$ and

$$(V_s^2) \int_a^b f(t)G(t)\, dt = G(a) \int_a^b f(t)\, dt$$

$$- (F(b)G'(b) - F(a)G'(a)) + \int_a^b F(t)G''(t)\, dt$$

where the latter integral is a Lebesgue integral.

PROOF. As before let H be defined as in (38). The corollary will follow from the theorem once we prove that $\Lambda_H(a, b)$ exists and establish that

(44) $$\Lambda_H(a, b) = G(a)\Lambda_F(a, b) - (F(b)G'(b) - F(a)G'(a)).$$

An integration by parts shows that

$$H(x) = F(x)G(x) - F(a)G(a) - 2\int_a^x F(t)G'(t)\, dt.$$

It follows that

$$\Lambda_H(a, b) = \Lambda_{FG}(a, b) - 2\left(F(b)G'(b) - F(a)G'(a)\right).$$

and this together with equation (14) yields (44).

12   Burkill's integration by parts formula

The title of this section refers to an integration by parts formula for the (SCP)–integral which Burkill claimed for his integral but neglected to prove. In [11] he appeals to one of his earlier papers [9] saying that a proof can be constructed as for the CP–integral "with some modification of detail". Subsequently it was pointed out that the proof does not follow in the same way because (SCP)–major and minor functions are C–continuous only almost everywhere while, to adapt the proof, C–continuity would be needed except on a countable set. The difficulty surfaces in the analogue of [9, Lemma 2] and its corollary. For if SCD $F(x_0) = f(x_0)$, $G'(x) = g(x)$, and $g(x)$ is bounded near $x = x_0$ then it is not true that, for $x = x_0$,

$$\underline{\text{SCD}}\, Fg \geq Fg + fG.$$

An example of Lee [30] shows this. Take $F(x) = x^{-1/2}$ for $x > 0$, $F(x) = (-x)^{-1/2}$ for $x < 0$, $F(0) = K$ and $G(x) = -x$. Then

$$\text{SCD}\, FG(0) = -\infty \neq K = F(0)G'(0) + G(0)\text{SCD}\, F(0).$$

This error plays no important role in the original paper since the main theorem on the representation of trigonometric series can be proved by using the theory of the formal multiplication of trigonometric series rather than appealing, as Burkill did, to an integration by parts formula. (Burkill would certainly have been aware of this since he modeled his paper after the article [34] where this method had been used.) Even so the problem of Burkill's integration by parts formula has somewhat plagued several specialists in the area. In 1980 Skljarenko [39] proved that Burkill's original formula is valid under the original hypotheses in [11] but the proof is long and the calculations are very complicated.

We can simplify the calculations to some degree by making the integration by parts formula follow as a consequence of the Mařík formula; this we shall present below. As a start towards this notice first that by making a stronger assumption than that used in [11] or [39] we obtain a simple version of the integration by parts formula for the (SCP)–integral.

THEOREM 68 *Let f be (SCP)–integrable on $[a, b]$, let F_1 be its (SCP)– primitive and let F be an indefinite integral of F_1. Suppose that G' is Lipschitz on $[a, b]$ and that*

$$(45) \qquad \limsup_{h \to 0+} |\mathrm{M}_s^2 F(x, h)| < +\infty$$

at nearly every point of (a, b). Then fG is (SCP)–integrable on $[a, b]$ and

$$(46) \quad (\text{SCP}) \int_a^b f(t)G(t)\, dt = F_1(b)G(b) - F_1(a)G(b) - \int_a^b F_1(t)G'(t)\, dt$$

where the latter integral exists in the Denjoy-Perron sense.

PROOF. Recall that, under the assumptions here, F is ACG_*, $F'(a) = F_1(a)$ and $F'(b) = F_1(b)$. As in Theorem 66 let

$$H(x) = \int_a^x G(t)\, dF(t) - \int_a^x F(t)\, dG(t).$$

We show that $\Lambda_H(a, b)$ exists. Note first that $H = FG - 2 \int F\, dG = FG - 2 \int FG'$ by an integration by parts. Then, since all the derivatives must exist,

$$(47) \qquad \Lambda_H(a, b) = (FG)']_a^b - 2\, FG']_a^b = F'G - FG']_a^b$$

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also exists. Also an integration by parts shows that

$$\int_a^b F(t)G''(t)\,dt = \int_a^b F(t)\,dG'(t)$$

(48)
$$= FG']_a^b - \int_a^b G'(t)F_1(t)\,dt$$

where the latter integral must be interpreted in the Denjoy-Perron sense.

Using Theorem 66 we have the formula

$$(\mathrm{V}_s^2)\int_a^b f(t)G(t)\,dt = \Lambda_H(a,b) + \int_a^b F(t)G''(t)\,dt.$$

Because of (47) and (48) this immediately supplies (46) at least if the first integral is interpreted as a V_s^2–integral. The proof is complete as soon as we see that it is in fact an (SCP)–integral. The second order primitive function is

$$K(x) = H(x) + \int_a^x \int_a^s F(t)G''(t)\,dt\,ds.$$

We need to show that K is ACG_* and that the one-sided derivatives $K'_+(a)$ and $K'_-(b)$ exist. The existence of the derivatives is clear from this and the formula $H = FG - 2\int FG'$ given above; since F is assumed to be a second primitive for f it has one-sided derivatives at a and b and G is continuously differentiable. From the same relation we see that FG, as a product of ACG_* functions, must be ACG_* too; it follows that H and hence also K are ACG_*. Thus fG is (SCP)–integrable on $[a,b]$ and the integral has the value $K'_-(b) - K'_+(a)$ which provides the desired integration by parts formula.

Let us now turn to the more difficult task of obtaining the formula (46) without using the condition (45). The proof depends on a series of computations given in Skljarenko [39] which we reproduce here without proofs. The first two are relatively elementary and straightforward. It is the third lemma (Lemma 71) that contains the deepest work; its proof takes five pages in [39, pp. 573–577].

LEMMA 69 *Let F be ACG_* on an interval $[a,b]$. Then there is a sequence of closed sets $\{P_n\}$ covering $[a,b]$ and a sequence of monotonic, absolutely continuous functions $\{m_k\}$ such that*

$$|F(x+h) - F(x)| \le |m_k(x+h) - m_k(x)| \quad (x \in P_k, x+h \in (a,b)).$$

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PROOF. See [39, Lemma 1, p. 568].

LEMMA 70 *Let F be a P^2–primitive on $[a,b]$. Then there is a sequence of closed sets $\{P_n\}$ covering $[a,b]$ and a sequence of smooth, convex functions $\{\tau_k\}$ such that*

$$|\Delta_s^2 F(x,h)| \le \Delta_s^2 \tau_k(x,h) \quad (x \in P_k, x+h \in (a,b)).$$

PROOF. See [39, Lemma 4, p. 573].

LEMMA 71 *Let F be continuous, let m be monotonic and absolutely continuous and let τ be smooth and convex on $[a,b]$. Suppose that for a closed set $P \subset [a,b]$*

$$|F(x+h) - F(x)| \le |m(x+h) - m(x)| \quad (x \in P,\ x+h \in (a,b))$$

and

$$|\Delta_s^2 F(x,h)| \le \Delta_s^2 \tau(x,h) \quad (x \in P,\ x+h \in (a,b)).$$

Then there is a nondecreasing, absolutely continuous function m_1 such that

$$|h(F(x+h) - F(x))| \le \int_0^h (m_1(x+t) - m_1(x-t))\, dt$$

for $x \in P,\ x+h \in (a,b)$.

PROOF. See [39, Lemma 5, p. 573].

   With these lemmas we can now prove the following lemma which is the key to applying the Mařík theory here.

LEMMA 72 *Let F be ACG_* and a P^2–primitive on $[a,b]$. Write*

$$\xi(x,h) = \mathrm{M}_s^2 F(x,h) h^2.$$

Then $\xi^*(E) = 0$ for every set $E \subset (a,b)$ with $|E| = 0$.

PROOF. Under these hypotheses we can apply Lemmas 69, 70 and 71 to obtain a sequence of closed sets $\{P_k\}$ covering $[a,b]$ and a sequence of non-decreasing, absolutely continuous functions $\{m_k\}$ such that

$$(49) \qquad |h(F(x+h) - F(x-h))| \le \int_0^h (m_k(x+t) - m_k(x-t))\, dt$$

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for $x \in P_k$, $x + h \in (a, b)$.

We write $\xi(x, h) = \frac{1}{2}\xi_1(x, h) + \xi_2(x, h)$ where

$$\xi_1(x, h) = h(F(x + h) - F(x - h))$$

and

$$\xi_2(x, h) = -\int_0^h (F(x + t) - F(x - t))\, dt.$$

Write $\tau_k = \int m_k$. The inequality (49) provides $\xi_1(x, h) \leq \Delta_s^2 \tau_k(x, h)$ for $x \in P_k$. Consequently

$$\xi_1^*(E \cap P_k) \leq \Delta_s^2 \tau_k^*(E \cap P_k)$$

for any set E. But τ_k is the integral of a monotonic, absolutely continuous function and so if $|E| = 0$ then, by the measure theory of Section 6, $\tau_k^*(E) = 0$. Thus $\xi_1^*(E \cap P_k) = 0$ for every set $E \subset (a, b)$ with $|E| = 0$. Hence $\xi_1^*(E \cap P_k) = \sum_{k=1}^{\infty} \xi_1^*(E \cap P_k) = 0$ also.

In a similar way it may be shown that $\xi_2^*(E) = 0$; here the proof is simpler still and needs only an appeal to Lemma 69. Hence $\xi^*(E) = 0$ and the lemma is proved.

We are now able to state and prove the main theorem of this section; the Burkill-Skljarenko integration by parts formula for the (SCP)–integral.

THEOREM 73 *Suppose that f is (SCP)–integrable on an interval $[a, b]$ with an (SCP)–primitive F_1 and suppose that G' is Lipschitz on $[a, b]$. Then fG is (SCP)–integrable on $[a, b]$ and the integration by parts formula (46) holds.*

PROOF. Let F be an indefinite integral of F_1 and write

$$\xi(x, h) = \mathrm{M}_s^2 F(x, h) h^2$$

and

$$\xi_1(x, h) = 2G'(x)\mathrm{M}_s^2 F(x, h) h^2 = 2G'(x)\xi(x, h).$$

If E is the set of points at which F' does not exist then $|E| = 0$ and so, by Lemma 72, $\xi^*(E) = 0$. As G' is bounded it follows easily that $\xi_1^*(E) = 0$ too.

The remainder of the proof is now identical with that for Theorem 66 except now we use the observations above to handle the exceptional set where F' does not exist.

13 An application to trigonometric series

Recall the following standard summability method. For any trigonometric series

(50) $$a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

the Riemann function for the series is defined as

(51) $$F(x) = a_0 x^2/4 - \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)/n^2$$

obtained by formally integrating the series twice. For example if the coefficients in the original series (50) are bounded the series (51) converges uniformly and so the Riemann function exists and is continuous. On the assumption that this function exists then the number c is said to be the sum of the original series by the Riemann method if $D_2 F(x) = c$. If the series (50) is convergent in the ordinary sense and has sum c then the Riemann method gives the same value (see [46, Vol I, p. 319]).

Our theorem asserts that, under certain broad conditions, a series will be Riemann summable almost everywhere and will be the (V_s^2)–Fourier series of its sum. In this form it is properly attributed to Mařík [35] and includes a number of similar theorems by Burkill, James and Marcinkiewicz and Zygmund. If the partial sums of a trigonometric series are bounded at *every* point (rather than at nearly every point as here) then the Riemann function for the series is smooth. The idea of allowing a countable exceptional set but assuming that the Riemann function is smooth is due to Zygmund (see his remark in [46, Vol II, p. 91]).

THEOREM 74 (Mařík) *Suppose that the partial sums of the trigonometric series (50) are bounded at nearly every point and that the Riemann function for the series is smooth. Then the series has a finite sum $f(x)$ by the Riemann method of summation at almost every point x, f is (V_s^2)–integrable on any period $[a, a + 2\pi]$ and the series is the Fourier series for f in this sense, i.e. for each n*

$$\pi a_n = (V_s^2) \int_0^{2\pi} f(t) \cos nt \, dt$$

and

$$\pi b_n = (V_s^2) \int_0^{2\pi} f(t) \sin nt \, dt.$$

PROOF. (cf. [35, (126), p. 74]) Since the partial sums are n.e. bounded the coefficients of the series must be bounded (see [46, Vol. I, p. 317]). Thus the function

$$F_1(x) = \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)/n^2$$

is defined everywhere and continuous. Clearly $F(x) = a_0 x^2/4 - F_1(x)$ is the Riemann function for (50). Note that F_1 is smooth (by hypothesis) and so, by (12), $\Lambda_{F_1}(a, a + 2\pi) = 0$ for any a. At every point x where the partial sums of the series (50) are bounded we know that $\Delta_s^2 F_1(x, h) = O(h^2)$ (see [46, Vol. I, p. 320]). Consequently F_1 has a second symmetric derivative $D_2 F_1(x) = f_1(x)$ that exists almost everywhere (by Theorem 48).

Translating these facts to the Riemann function we have that F is continuous, smooth, $\Lambda_F(a, a + 2\pi) = a_0 \pi$, $\Delta_s^2 F(x, h) = O(h^2)$ nearly everywhere and $D_2 F(x) = a_0/2 + f_1(x) = f(x)$ exists almost everywhere. By definition, the series has a finite sum $f(x)$ by the Riemann method of summation at almost every point x and (by Theorem 48) f is V_s^2–integrable over any period with

$$(V_s^2) \int_0^{2\pi} f(t) \, dt = \pi a_0.$$

This gives the Fourier formula for a_0.

The remaining coefficients we obtain from the integration by parts formula of Theorem 66. Fix a natural number m and write $G(x) = \cos mx$. Note that G' is Lipschitz, $G(0) = G(2\pi)$ and $\Lambda_{F_1}(0, 2\pi) = 0$. By Theorem 5 $M_s^2 F_1(x, h)$ is bounded. This supplies all the conditions needed to obtain the integration by parts formula

$$\int_0^{2\pi} f_1(t) G(t) \, dt = \int_0^{2\pi} F_1(t) G''(t) \, dt = a_m \pi.$$

Applying this to f we obtain

$$(V_s^2) \int_0^{2\pi} f(t) \cos mt \, dt = \int_0^{2\pi} (f_1(t) + a_0/2) G(t) \, dt = \pi a_m.$$

The formulae for the Fourier sine coefficients are similarly obtained and so the theorem is proved.

Note that while the above theorem has been expressed for the V_s^2–integral it can be reformulated for the narrower (SCP)–integral because of the fact that under these hypotheses the once integrated series is a Lebesgue–Fourier

series; if F is the Riemann function then it is the Lebesgue integral of F' and so even absolutely continuous. We can equally use the (P^2)–integral but then the formula for the coefficients has to be reformulated. The next two corollaries express these observations.

COROLLARY 75 *Under the same conditions as the theorem there is a set of full measure B so that $b + 2\pi \in B$ whenever $b \in B$, f is (SCP)– integrable on any interval $[a, b]$ (a, $b \in B$) and the coefficients of the series are determined by the formulae*

$$\pi a_n = (\text{SCP}) \int_b^{b+2\pi} f(t) \cos nt \, dt \quad , \quad \pi b_n = (\text{SCP}) \int_b^{b+2\pi} f(t) \sin nt \, dt$$

for any $b \in B$.

PROOF. If F is the Riemann function for (50) then F is absolutely continuous. Write $B_1 = \{x; \ F'(x) \text{ exists}\}$ and $B = \bigcap_{n=1}^\infty B_1 - 2n\pi$. By Theorem 60 the V_s^2–integral reduces to the (SCP)–integral on any interval $[a, b]$ (a, $b \in B$).

COROLLARY 76 *Under the same conditions as the theorem f is (P^2)– integrable on any interval and the coefficients of the series are determined by the formulae*

$$a_n = -\frac{1}{\pi^2} \int_{-2\pi,2\pi,0} f(t) \cos nt \, dt \quad , \quad b_n = -\frac{1}{\pi^2} \int_{-2\pi,2\pi,0} f(t) \sin nt \, dt.$$

PROOF. (cf. [46, Vol. II, pp. 89–90]) If F is the Riemann function for (50) then by the theorem $\Delta_s^2 F = f(\Delta \ell)^2$ on any interval. The formula for the coefficients now follows from Corollary 56 relating the V_s^2–integral to the (P^2)–integral.

A further corollary allows one to integrate a trigonometric series. Recall that the *Lebesgue function* of a trigonometric series (50) is the once integrated series

(52) $$L(x) = a_0 x/2 + \sum_{n=1}^\infty \frac{a_n \sin nx - b_n \cos nx}{n}.$$

COROLLARY 77 *Suppose that the partial sums of the trigonometric series (50) are bounded at nearly every point and that a_n, $b_n \to 0$. Let B denote the set of points at which the Lebesgue function (52) for the series (50) exists. Then the series has a finite sum $f(x)$ by the Riemann method of summation at almost every point x, B has full measure in any interval and f is (V_s^2)–integrable (and (SCP)–integrable) on any interval $[a, b]$ $(a, b \in B)$. The integral can be expressed as*

$$(\mathrm{V}_s^2) \int_a^b f(t)\, dt = a_0(b - a)/2 + \sum_{n=1}^{\infty} \frac{a_n(\sin nb - \sin na) - b_n(\cos nb - \cos na)}{n}.$$

PROOF. Since a_n, $b_n \to 0$ the Riemann function for the series is smooth. The existence of the Lebesgue function $L(x)$ a.e. is well known ([46, Vol. I, p. 321]). Now if F is the Riemann function for (50) then $\Lambda_F(a, b) = F'(b) - F'(a) = L(b) - L(a)$ (see, for example, [34, Lemma 31, p. 41]) and the corollary follows.

The final corollaries are stated so as to display conditions under which the assumed smoothness of the Riemann function may be dropped. In each case the conditions are known to be sufficient in order that the Riemann function be smooth everywhere. The needed material may be found in [35, pp. 76–81] and [46, Vol. I, Chap. 9, 11]. See also Zygmund's remark in [46, Vol. II, p. 91].

COROLLARY 78 *Suppose that the partial sums of the trigonometric series (50) are bounded at every point. Then the series is the V_s^2–Fourier series for its Riemann sum.*

COROLLARY 79 *Suppose that the partial sums of the trigonometric series (50) are bounded at nearly every point and*

$$\sum_{k=1}^{n} k\sqrt{a_k^2 + b_k^2} = o(n)$$

as $n \to +\infty$. Then the series is the V_s^2–Fourier series for its Riemann sum.

COROLLARY 80 *Suppose that the trigonometric series (50) converges at nearly every point. Then the series is the V_s^2–Fourier series for its sum.*

The conditions given in the theorem are not sufficient as they stand to guarantee that the Riemann function is necessarily smooth. Mařík [35, pp.76–81] (cf. also [46, Vol. II, p. 91]) supplies the example $\sum_{n=1}^{\infty} \cos nx$ to show that the smoothness of F must be assumed in the statement of the theorem. The Riemann function for this series is not smooth but it is particularly simple,

$$F(x) = -\frac{x^2}{4} + \frac{\pi x}{2} - \frac{\pi^2}{6} \qquad (0 \leq x \leq 2\pi).$$

Clearly the series is not a V_s^2–Fourier series of F''. Here

$$\sup_{N} \left| \sum_{n=1}^{N} \cos nx \right| < +\infty$$

for $0 < x < 2\pi$ and the series satisfies all the other hypotheses of the theorem. Curiously the theorem does apply to the conjugate series; in fact $\sum_{n=1}^{\infty} \sin nx$ is the V_s^2–Fourier series of $f(x) = \frac{1}{2} \cot \frac{x}{2}$ $(0 < x < 2\pi)$. In this case even though f is not Lebesgue integrable the formulae for its Fourier coefficients uses only the Lebesgue integrable functions $f(x) \sin nx$ and so this is a generalized Fourier sine series in the sense of Zygmund [46, Vol. I, p. 48].

14 Some further applications

The theory of the symmetric integrals allows a number of classical results in the study of trigonometric and Fourier series to be interpreted in a particularly simple way. Both of the theorems in this section are relatively routine applications of the integration theory that we have presented. Of course these theorems can be proved without developing quite this much technical apparatus but they become more transparent and natural in this setting. For example the first theorem (Theorem 81) originally due to W. H. Young can be proved from the fact that a continuous, smooth function with a second symmetric derivative nearly everywhere zero must be linear; the proof here merely uses the V_s^2–Fourier coefficients.

THEOREM 81 (Young) *Suppose that the trigonometric series (50) converges nearly everywhere to zero. Then the coefficients must all vanish.*

PROOF. Let $f(x)$ denote the sum of the series. By Theorem 74 the series is the V_s^2–Fourier series for f and so the coefficients must vanish.

THEOREM 82 (de la Vallée Poussin) *Suppose that the trigonometric series (50) has bounded partial sums $s_n(x)$ at nearly every point and that a_n, $b_n \to 0$. Write $\overline{s}(x) = \limsup_{n\to\infty} s_n(x)$ and $\underline{s}(x) = \liminf_{n\to\infty} s_n(x)$. If both $\overline{s}$ and $\underline{s}$ are Lebesgue [Denjoy–Perron] integrable then the series is the Fourier series of $f = D_2 F$ in that sense where F is the Riemann function for the series.*

PROOF. (cf. [46, Vol. I, Theorem (3.19), p. 328]) By Theorem 74 the series is the V_s^2–Fourier for $f = D_2 F$ which exists almost everywhere. By [46, Vol. I, Theorem (2.7), p. 320]) $f(x)$ is almost everywhere contained between the values

$$\frac{1}{2}(\overline{s}(x) + \underline{s}(x)) \pm \frac{1}{2}k(\overline{s}(x) - \underline{s}(x))$$

for some k and so f must be integrable in the same sense in which these functions are integrable.

Note that the hypothesis that a_n, $b_n \to 0$ in the statement of the theorem can be replaced by the assumption that the Riemann function F is everywhere smooth. A corollary expresses a useful special case (cf. [46, Vol. I, Theorem (3.18), p. 328]).

COROLLARY 83 *Suppose that the trigonometric series (50) converges nearly everywhere to a function f. If $f \geq g$ where g is Lebesgue [Denjoy–Perron] integrable then the series is the Fourier series for f in that sense.*

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George E. Cross,
Department of Pure Mathematics,
University of Waterloo,
Waterloo, Ontario, Canada N2L 3G1.

Brian S. Thomson,
Department of Mathematics,
Simon Fraser University,
Burnaby, B.C.,
Canada V5A 1S6.