MR2202919 26A42 26A24 26A39

Hagood, John W. (1-NAZ-MS); **Thomson, Brian S.** (3-SFR)

Recovering a function from a Dini derivative.

Amer. Math. Monthly **113** (2006), *no.* 1, 34–46.

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MR2083831 (2005g:26014) 26A45

Thomson, Brian S. (3-SFR)

Vitali coverings and Lebesgue's differentiation theorem. (English. English summary)

Real Anal. Exchange **29** (2003/04), *no.* 2, 957–972.

The author gives a new proof of the well-known Lebesgue differentiation theorem, by considering arbitrary functions which possess the Vitaly property instead of considering monotonic functions. He also proves that every continuous monotonic function has the Vitaly property on every Borel set. Some characterizations and criteria of the Vitaly property are also given.

Živorad Tomovski (FMD-SKOPN-IM)

MR1954615 (2004b:28008) 28A15 26A24

Thomson, Brian S. (3-SFR)

Differentiation.

Handbook of measure theory, *Vol. I*, *II*, 179–247, *North-Holland*, *Amsterdam*, 2002.

Differentiation is a vast subject and it's no wonder that Brian Thomson's survey paper is the longest (68 pages) among the articles of the *Handbook of measure theory*. The central theme of the paper around which the material is arranged is that of derivation bases. Abstract differentiation theory is discussed by several monographs. Unfortunately, they "require a serious devotion to a viewpoint and an elaborate language in order to enter their universe" as Thomson puts it. In other words, they offer abstract schemes that are difficult to follow and do not seem to be rewarding enough.

In the first half of the paper Thomson offers a fairly simple and rather general theory of abstract differentiation. Its basic notion is the covering relation which is simply a set of pairs (I,x), where I is a subset and x is an element of a given set X. (In the simplest special case we put $X=\mathbb{R}$ and take those pairs (I,x) where I is an interval and $x\in I$.) By a derivation basis we mean a collection $\mathcal{B}$ of covering relations satisfying certain axioms. These axioms express the condition that the basis is a filter, has a local structure, and is compatible with the topology on X if there is given any. (In the simplest special case of the so-called interval basis we take the collection of covering relations $\{(I,x)\colon |I|<\delta,\ x\in I\}$ for every $\delta>0$.)

If a function h is defined on the set of pairs that occur in the covering relations of the basis then its $\mathcal{B}\limsup$ at a point x is defined as $\inf_\beta \sup_{(I,x)\in\beta} h(I,x)$, where β runs through all covering relations of the basis. (If h is an interval function then, in the case of the interval basis we obtain the upper derivative of h.) The $\mathcal{B}\liminf$ of h is defined analogously.

A basic ingredient of the theory is the dual basis $\mathcal{B}^*$ defined in a natural way. (In the case of the interval basis the dual basis consists of those covering relations that correspond to all Vitali coverings of $\mathbb{R}$.) Other notions to be introduced are the variation of h with respect to the basis and the outer measure induced by the variation.

Now let X be a metric space, let μ be a locally finite Borel measure on X, and suppose that for every pair (I,x) of the covering relations of a given basis the set I is a bounded Borel set with $0<\mu(I)<\infty$. Let f be integrable with respect to μ, and define $h(I,x)=\frac{1}{\mu(I)}\int_I|f(t)-f(x)|\,d\mu(t)$. The main result (Theorem 41) of the theory offered by Thomson states that $\mathcal{B}\lim h=0$ holds μ_*-almost everywhere, where μ_* is the outer measure induced by the variation of μ with respect to the dual basis $\mathcal{B}^*$.

This remarkable theorem gives the strongest possible result using the weakest possible conditions. We have to bear in mind, however, that under these very general conditions nothing guarantees that $\mu_*=\mu$ or $\mu_*\neq 0$. If, for example, $X=\mathbb{R}^2$ and $\mathcal{B}$ is the set of covering relations β_δ where $(I,x)\in\beta_\delta$ if I is a rectangle of diameter $<\delta$ and $x\in I$, then $\lambda_*\equiv 0$, since, by a classical example due to H. Bohr, the $\mathcal{B}$-derivative of the integral of an integrable function does not exist in general.

It is the question whether or not $\mu_*=\mu$ holds, where the geometry of the basis $\mathcal{B}$ enters into the discussion. Thomson gives a review of those conditions that imply this equality and, consequently, the differentiation of integrals. In this part of the paper he gives a survey of the literature of the density property, Vitali type covering theorems, net bases, halo properties, the (Q)-property and the Besicovitch-Morse property.

Other topics closely related to differentiation are also discussed. These include the integration of derivatives (Henstock-Kurzweil integral and its variants), the symmetric derivative, unusual density bases on $\mathbb{R}$, the approximate derivative, the theorem of de la Vallée Poussin and the Radon-Nikodým theorem. The list of references of this highly readable and fascinating article contains 91 items.

{For the entire collection see MR1953489 (2003h:28001)}

M. Laczkovich (H-EOTVO-AN)

MR1876254 (2003e:01030) 01A70 01A60

Bruckner, Andrew M. (1-UCSB); **Thomson, Brian S.** (3-SFR)

Real variable contributions of G. C. Young and W. H. Young.

Expo. Math. **19** (2001), *no.* 4, 337–358.

The authors discuss some of the contributions of W. H. and G. C. Young to real variable theory. They point out that most of the papers attributed to W. H. Young alone were in fact joint work with Grace.

Among the topics discussed are: inner limiting sets, now known as G_δ sets, and the classification of sets arising in analogous ways; questions about the derivates of functions in a given Baire class; the unsolved problems of the characterisation of derivatives and whether the product of two derivatives is a derivative; and the Denjoy-Young-Saks theorem. The authors also describe later work by other authors. The Youngs' work on differentials and semicontinuous functions is mentioned.

The authors seem to have concentrated on the point-set aspects of real-variable theory; some other work in the field is omitted; for instance, the work on integration theory does not appear, nor does their joint expository paper of 1912 on variants of the Riesz-Fischer theorem. *F. Smithies* (Cambridge)

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MR1778525 (2001h:26014) 26A39 26A45

Thomson, Brian S. (3-SFR)

The space of Denjoy-Perron integrable functions. (English. English summary)

Real Anal. Exchange **25** (1999/00), *no.* 2, 711–726.

Let $\{E_n\}$ be an increasing sequence of closed sets covering a fixed interval $[a,b]$ of the real line. $\mathcal{D}P[a,b]$ denotes the space of all Henstock-Kurzweil integrable functions $f\colon[a,b]\to\mathbf{R}$, and $\mathcal{D}P(\{E_n\})$ the space of all functions $f\in\mathcal{D}P[a,b]$ such that the primitive F of f is BV_* (in the Saks sense) on each set E_n.

In this paper it is proved that: (1) the sequence of seminorms $p_n(f)=\mathrm{Var}(F,E_n)$ defines on $\mathcal{D}P(\{E_n\})$ a metrizable, complete, locally convex topology $\mathcal{T}(\{E_n\})$; (2) L^∞ is the dual of $\mathcal{D}P(\{E_n\})$ endowed with the topology $\mathcal{T}(\{E_n\})$; (3) the Alexiewicz norm topology on $\mathcal{D}P[a,b]$ is the finest convex topology such that each of the canonical injections from the spaces $\mathcal{D}P(\{E_n\})$ into $\mathcal{D}P[a,b]$ is continuous. *B. Bongiorno* (I-PLRM)

MR1704758 (2000g:26006) 26A45 28A12

Thomson, Brian S. (3-SFR)

Some properties of variational measures. (English. English summary)

Real Anal. Exchange **24** (1998/99), *no.* 2, 845–853.

Let F be a nonnegative interval function and let $E\subset[a,b]$. The gauge variation of F on E is the infimum over gauge δ of $\sup\{\sum\tau(a_i,b_i)\}$, where the supremum is taken over all disjoint collections $\{(a_i,b_i)\}$ of open subintervals of (a,b) for which there is a point $\xi_i\in E\cap(a_i,b_i)$ satisfying the condition $b_i-a_i<\delta(\xi_i)$.

In this paper it is proved that if the gauge variation of F is σ-finite on all closed subsets of E that have zero Lebesgue measure, then it is σ-finite on E. *B. Bongiorno* (I-PLRM)

MR1691755 (2000g:28029) 28C10 26A30

Shi, Hongjian (3-SFR)**; Thomson, Brian S.** (3-SFR)

Haar null sets in the space of automorphisms on $[0,1]$**.**
(English. English summary)

Real Anal. Exchange **24** (1998/99), *no.* 1, 337–350.

Let G be an arbitrary Polish group. A Borel probability measure μ on G is called left [resp., right] transverse to a universally measurable subset X of G provided $\mu(gX)=0$ [resp., $\mu(Xg)=0$] for all g in G. The authors give examples to show that these two notions are independent in the group $\mathcal{H}[0,1]$ of all homeomorphisms h of $[0,1]$ with $h(0)=0$ and $h(1)=1$, where the group operation is the composition of functions and the topology is that of uniform convergence. Moreover, answering a question of Jan Mycielski (1992), they define a Borel subset X of $\mathcal{H}[0,1]$ such that there is a Borel probability measure μ on $\mathcal{H}[0,1]$ which is both left and right transverse to X, but there is no Borel probability measure ν on $\mathcal{H}[0,1]$ with $\nu(gXh)=0$ for all g,h in G. The latter means that X is not Haar null in the sense of J. P. R. Christensen (1974).

An announcement of the results in this version of the paper appears in the same volume of the journal in a report of a conference [Real Anal. Exchange **24** (1998/99), no. 1, 113–116].

{Reviewer's remarks: (1) In the definition of g on p. 343, "I" should be replaced by another subinterval of $[0,1]$. The definition of g on p. 344 requires a similar modification. (2) The representation of S on p. 345 seems wrong. One should use there the sets $(0,2^{-n}]\cap\mathbf{Q}$, rather than $\{2^{-n}\colon\ n=m,m+1,\cdots\}$.} *Z. Lipecki* (PL-PAW)

MR1609830 (99f:28005) 28A12 26A46

Thomson, Brian S. (3-SFR)

σ**-finite Borel measures on the real line. (English. English summary)**

Real Anal. Exchange **23** (1997/98), *no.* 1, 185–192.

Let f be an ACG$_*$ function on $[a,b]$ in the sense of S. Saks [*Theory of the integral*, Second revised edition. English translation by L. C. Young. With two additional notes by Stefan Banach, Dover, New York, 1964; MR0167578 (29 #4850)(§8, Chapter VII)]. Given $\varnothing\neq E\subset[a,b]$ denote by $\mathcal{G}(E)$ the class of all strictly positive, finite functions on E and, for $\delta\in\mathcal{G}(E)$, put

$$V(f,E,\delta)=\sup\sum|f(b_i)-f(a_i)|,$$

where the sup is taken over all collections of nonoverlapping intervals

$[a_i, b_i] \subset [a, b]$ such that there is an $x_i \in E \cap [a_i, b_i]$ with $0 < b_i - a_i < \delta(x_i)$. Defining $\mu^*_f(\varnothing) = 0$ and $\mu^*_f(E) = \inf\{V(f, E, \delta);\ \delta \in \mathcal{G}(E)\}$ for $\varnothing \neq E \subset [a, b]$, one arrives at a metric outer measure μ^*_f whose restriction to the σ-algebra of Borel subsets of $[a, b]$ is a Borel measure μ on $[a, b]$ which is shown to enjoy the following properties: (1) There is a sequence of closed sets $F_n \subset [a, b]$ such that $\mu(F_n) < \infty$ for each n and $\bigcup F_n = [a, b]$; (2) μ is absolutely continuous with respect to Lebesgue measure; (3) $\mu(B) = \mu^*_f(B) = \int_B |f'(x)|dx$ for every Borel set $B \subset [a, b]$. Conversely, if a Borel measure μ on $[a, b]$ satisfies (1),(2), then there is an ACG$_*$ function f on $[a, b]$ such that (3) is valid, too. *J. Král* (CZ-AOS)

MR1610467 (99c:26004) 26A24 26A42

Freiling, C. (1-CASSB); **Rinne, D.** (1-CASSB);
Thomson, B. S. (3-SFR)

A Riemann-type integral based on the second symmetric derivative. (English. English summary)

J. London Math. Soc. (2) **56** (1997), *no.* 3, 539–556.

If F is continuous and at each point x in an interval $f(x) = \mathrm{SD}_2F(x) = \lim_{h\to 0}(F(x+h) + F(x-h) - 2F(x))/h^2$ exists, the problem motivating this article is the following: How can one recover $F(x)$ using a Riemann-style integral? Alternatively, if $K = [a, b]$ and for $0 < p \leq (a+b)/2$, how can one recover $\Delta K_pF = F(a) + F(b) - F(a+p) - F(b-p)$ using an integral involving partition sums? The solution is a second-order integral defined on "2-intervals" $K_p = ([a, b], [a+p, b-p])$ involving a gauge, an exceptional set E and a second gauge on $E \times \mathbf{N}$, so that for partitions using "regular" 2-intervals one obtains the integral $I(f, K_p)$ as a limit of partition sums. Several elaborate, but straightforward, partitioning arguments for rectangles in the plane are needed and occur at the beginning of the paper along with a covering theorem guaranteeing the existence of the required partitions. These are in turn needed to guarantee the uniqueness of the integral (and its properties). A first-order integral $\int_a^b f$ is given by $\lim_{p\to 0+} I(f, ([a, b], [a+p, b-p]))/p$ providing the limit exists. This integral is more general than the Riemann-complete integral and exists and can be used to determine the Fourier coefficients of an everywhere convergent trigonometric series when given the limit function $f(x)$. This, as is asserted at the beginning of the paper, is the problem which kept SD_2F "as an object of study for nearly a century and a half". *James Foran* (1-MOKC)

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Note: This list, extracted from the PDF form of the original paper, may contain data conversion errors, almost all limited to the mathematical expressions.

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MR1426684 (97j:26008) 26A39

Bongiorno, B. (3-SFR); **Pfeffer, W. F.** (3-SFR); **Thomson, B. S.** (3-SFR)

A full descriptive definition of the gage integral. (English. English summary)

Canad. Math. Bull. **39** (1996), *no.* 4, 390–401.

A Kurzweil-Henstock-type sum integrability of a function over a compact nondegenerate interval (called a cell) in $\mathbf{R}^n$ is defined.

Necessary and sufficient conditions are given for a continuous F defined on the family of all subcells of a cell $A \subset \mathbf{R}^n$ to be derivable to F' almost everywhere in A and such that F is the indefinite integral of F'. These conditions are given in terms of the critical or essential critical variation of F and the fact that F belongs to AC_*. In this way a generalization of the descriptive definition of the Denjoy-Perron integral to n-dimensional cells is presented. *Š. Schwabik* (Prague)

MR1407282 (97i:26004) 26A21 26A24

Freiling, C. (1-CASSB); **Thomson, B. S.** (3-SFR)

Scattered sets and gauges. (English. English summary)

Real Anal. Exchange **21** (1995/96), *no.* 2, 701–707.

As the abstract to this note promises: "An elementary and natural method for demonstrating that certain exceptional sets are scattered is presented." Recall that a set of real numbers is scattered if every nonempty subset has an isolated point. Likewise, a set is called right [left] scattered if every nonempty subset has a point isolated on the right [left]; any such set is called semi-scattered. Finally, a set is splattered if every nonempty subset has a point isolated on at least one side. Under various names (e.g., separierte Mengen, clairsemé, and zertreute Mengen) and sometimes with no name, scattered sets have found their way into the literature of real analysis for over a century, occurring as countable exceptional sets to some behavior. Here the authors provide the following tool: If δ is a gauge function (i.e., a function into $\mathbf{R}^+$) defined on all of $\mathbf{R}$ except possibly for some countable set, then, except for a right [left] scattered set, every point is the limit from the right [left] of some sequence $\{x_i\}$, for which $\delta(x_i)$ is bounded above zero. They then illustrate how this tool may be used to provide quite nice proofs of several known results involving exceptional sets which are scattered, semi-scattered, or splattered. These include results of T. Viola [Ann. École Norm. (3) **50** (1933), 71–125; Zbl 007.05901], Z. Charzyński [Fund. Math. **21** (1931), 214–225; Zbl 008.34401], M. J. Evans and L. M. Larson [Acta

Math. Hungar. **43** (1984), no. 3-4, 251–257; MR0733857 (85h:26005)], and Freiling [Trans. Amer. Math. Soc. **318** (1990), no. 2, 705–720; MR0989574 (90g:26003)].

{See also the following review [MR1407261 (97i:26005)].}

Michael Evans (1-WLEE)

MR1407265 (97g:26009) 26A39

Skvortsov, V. A. (RS-MOSC); **Thomson, B. S.** (3-SFR)

Symmetric integrals do not have the Marcinkiewicz property. (English. English summary)

Real Anal. Exchange **21** (1995/96), *no.* 2, 510–520.

One of the more surprising results in the Perron integral theory is the Marcinkiewicz theorem: a function is Perron integrable iff it has one pair of continuous major and minor functions. This result, which is in the classic book of Saks, has been extended to the CP-integral and to the AP-integral. However, in his unpublished thesis, Sklyarenko showed that the result is false for the SCP-integral; and later the first author showed that this is also the case for the dyadic Perron integral. The present interesting paper shows that the Marcinkiewicz theorem fails for all known Perron integrals defined using symmetric derivatives; that is, in each case there is a non-integrable function with a pair of continuous major and minor functions in the sense of the integral. The authors point out that in all cases where the theorem fails the associated derivative at a point does not use the function value at that point. The methods of proof use the idea of symmetric variation, details of which can be found in the book by the second author [*Symmetric properties of real functions*, Dekker, New York, 1994; MR1289417 (95m:26002)]. *P. S. Bullen* (3-BC)

MR1407261 (97i:26005) 26A21 54H05

Freiling, C. (1-CASSB); **Thomson, B. S.** (3-SFR)

Scattered sets, chains and the Baire category theorem. (English. English summary)

Real Anal. Exchange **21** (1995/96), *no.* 2, 440–458.

Whereas the article by the same authors reviewed immediately above [Real Anal. Exchange **21** (1995/96), no. 2, 701–707; MR1407282 (97i:26004)] provides a convenient tool for showing that an exceptional set in real analysis is scattered, semi-scattered, or splattered, this interesting companion article presents a much more in-depth analysis of the structure of such sets.

Fundamental to this analysis are the notions of a chain of open sets and its associated scattered set. A chain of open sets is a well-ordered, possibly transfinite, sequence $G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots$ of open subsets of $\mathbf{R}$. For each ordinal α, R_α [resp. L_α] denotes the set of finite right [resp. left] hand endpoints of components of G_α. Then $R = \bigcup_\alpha R_\alpha$ [resp. $L = \bigcup_\alpha L_\alpha$] is left [resp. right] scattered. The set R [resp. L] is called the associated left [resp. right] scattered set of the chain and $R \cap L$ [resp. $R \cup L$] is called the associated scattered [resp. splattered] set of the chain. The authors show that every set that is scattered [resp. left scattered, right scattered] is the associated scattered [resp. left scattered, right scattered] set of some chain. Actually, they establish an even more detailed analysis of the structure of the set in question and the above is a corollary. Characterizations of certain splattered sets are also given.

The authors proceed to show that any application of the Baire category theorem on the real line leads naturally to a chain of open sets and hence to an exceptional scattered set. Some applications of this "scattered Baire theorem" are provided. One interesting example is the following variation of the Cantor-Bendixson theorem: Every set can be partitioned into four pieces, the first scattered, the second left scattered and having no isolated points, the third right scattered and having no isolated points, and the fourth having only bilateral limit points. *Michael Evans* (1-WLEE)

MR1289417 (95m:26002) 26Axx 42A24

Thomson, Brian S. (3-SFR)

★**Symmetric properties of real functions. (English. English summary)**

Monographs and Textbooks in Pure and Applied Mathematics, 183.
Marcel Dekker, *Inc.*, *New York*, 1994. xvi+447 *pp.* $150.00.
ISBN 0-8247-9230-0

This book is designed to give the reader (who is presumed to know only the basics of Lebesgue theory) an almost complete picture of the subject: symmetric real analysis of functions. The text consists of comments, short insightful proofs and constructions; theorems and examples involving long proofs are either broken down into lemmas or are given a reference so the reader can pursue them in the literature. The author also provides a historical view of the subject matter and includes a large miscellany of results involving symmetric relations. An appendix with some material needed for the text, several pages of problems (49) and an extensive bibliography round out the text.

Generalizations of the ordinary first and second derivatives, the first and second symmetric derivatives, are

$$SDf(x) = \lim_{t\to 0}(f(x+t) - f(x-t))/2t \quad \text{and}$$

$$SD_2f(x) = \lim_{t\to 0}(f(x+t) + f(x-t) - 2f(x))/t^2.$$

A function is said to be symmetrically continuous at a point x if the numerator of $SDf(x)$ approaches 0 as t approaches 0; it is said to be symmetric at x if the numerator of $SD_2f(x)$ approaches 0 as t approaches 0. A natural question which occurs throughout the book is "To what extent do symmetric concepts behave like the ordinary ones?" Since the attention given to symmetric real analysis is largely due to applications to trigonometric series, problems involving such series are naturally a recurring theme. It is only possible, in a brief summary, to give a few of the many threads of which this book is woven.

In Chapter 1, two theorems of Riemann involving trigonometric series are given credit for originating and motivating the study of the second symmetric derivative. These lead naturally to theorems which give conditions guaranteeing the linearity or convexity of a function. The approximate symmetric derivative of f, $ASDf(x)$, is introduced (the symmetric derivative at x with respect to a set of t with density 1 at 0) and the chapter concludes with an important result of Khinchin: If f is measurable then f is differentiable at almost every point where $\overline{SD}f(x) < \infty$ and, in particular, at almost every point where $SDf(x)$ exists. Chapter 2 deals with symmetric continuity and symmetry and begins with results obtained in the first half of the 20th century. It is centered on a theorem of Charzyński which asserts that if $-\infty < \underline{SD}f(x) \leq \overline{SD}f(x) < \infty$ at every point of an interval, then the set of points of discontinuity of f is a scattered set; i.e., each nonempty subset of the set contains an isolated point (thus, it is also nowhere dense and at most countable). A variety of related results complete the chapter. Chapter 3 deals with covering theorems, theorems which assert that a specific type of cover of a set contains a certain type of subcover. The author studied these properties extensively in two earlier survey articles [Real Anal. Exchange **8** (1982/83), no. 1, 67–207; MR0694507 (84i:26008a); Real Anal. Exchange **8** (1982/83), no. 2, 278–442; MR0700194 (84i:26008b)]. Recent results of Freiling and Rinne along with those of the author and Preiss and others are presented. Such theorems are essential to monotonicity theorems, differentiation and the theory of integration (which is to follow in Chapter 9). Symmetry here means that a point x is contained in an interval of the form $[x-h, x+h]$ in a cover. Properties involving the numerator of SD_2f are called even properties and these are the subject of Chapter 4. Monotonicity theorems (theorems which assert sufficient conditions for a function to be monotone nondecreasing and which are important for integrals) are the subject of Chapter 5. The central result is the monotonicity theorem for the symmetric approximate derivative, which was only recently given a correct proof. Properties involving the numerator of $SDf(x)$ are called odd properties and form the subject of Chapter 6. Since these properties do not involve the value of the function f at the point x, they tend to be trickier and involve functions which are not measurable. Chapter 7 is a study of the symmetric derivative. While symmetrically differentiable functions need not be measurable, a surprising result of the author and Preiss is that the symmetric derivative of a function on an interval is necessarily measurable. Chapter 8 develops the notion of symmetric variation, which is used for the integrals developed in Chapter 9. For example, let $S_\delta(f,E)=\sum|f(x_i+h_i)-f(x_i-h_i)|$, where the sum is over sequences of nonoverlapping intervals with x_i in E; then the symmetric variation of f on E is $VS_f(E)=\sup S_\delta(f,E)$. In Chapter 9, this variation and the monotonicity theorem for the approximate symmetric derivative are used to produce an integral definable as a Riemann type integral. An integral based on ASD_2, capable of computing the Fourier coefficients of a function which has an even convergent Fourier series, is presented. Along with these results of the author and Preiss, Chapter 9 contains information on other integrals more general than that of Lebesgue. *James Foran* (1-MOKC)

MR1228433 (94i:26003) 26A24

Thomson, Brian S. (3-SFR)

The range of a symmetric derivative.

Real Anal. Exchange **18** (1992/93), *no.* 2, 615–618.

The author gives a simple proof of the result that there is no symmetrically differentiable function whose symmetric derivative assumes just two finite values, obtained by Z. Buczolich and M. Laczkovich [Acta Math. Hungar. **57** (1991), no. 3-4, 349–362; MR1139329 (92k:28002)]. He also proves that, given three real numbers α, β, γ, with $\alpha<\gamma<\beta$, $\gamma\neq\frac{1}{2}(\alpha+\beta)$, there is no symmetrically differentiable function whose symmetric derivative assumes just the three values α, β and γ. *Tapan Kumar Dutta* (Burdwan)

MR1192420 (94a:28008) 28A12 26A42

Pfeffer, Washek F. (1-CAD); **Thomson, Brian S.** (3-SFR)

Measures defined by gages. (English. English summary)

Canad. J. Math. **44** (1992), *no.* 6, 1303–1316.

A finitely additive volume $\nu(A) \geq 0$ is given for each set A in a family S of subsets of a locally compact Hausdorff space X, A having closure A^- and interior A°. We are given that if $A, B \in S$ then A^- is compact, $A \cap B \in S$, there are disjoint sets $C_j \in S$ $(1 \leq j \leq n)$ with union $A - B$, and for each $x \in X$, $\{A \in S\colon x \in A^\circ\}$ is a neighbourhood base at x. In the language of the reviewer [see, e.g., *The general theory of integration*, Oxford Univ. Press, New York, 1991; MR1134656 (92k:26011)], the authors construct a McShane-type division space and integrals of functions $f(x)\nu(I)$, for $x \in A^-$, $A \in S$, $I \in S$, but not necessarily $x \in I^-$, using the name "partition" instead of "division". The variation $\nu^*(E)$ of $\chi(E;x)\nu(I)$, for χ the characteristic function of $E \subset X$, is shown to be an outer measure of E with various properties. Gage measurability of certain sets $E \subset X$ is defined and proved to be equivalent to the classical definition that, given $\varepsilon > 0$, there are a closed set F and an open set G with $F \subset E \subset G$, $\nu^*(G - F) < \varepsilon$. For S^* the family of such $E \subset X$, and M the family of Carathéodory ν^*-measurable sets of X, then $S^* \subset M$, while if $E \in M$ and $\nu^*(E) < \infty$ then $E \in S^*$. If the measure from ν^* is σ-finite then $S^* = M$. If it is not σ-finite and if Σ is the family of all σ-finite sets in X, $\Sigma \subset S^*$ if X is metacompact (each open cover C^* of X has an open refinement C with $\{E \in C\colon x \in E\}$ finite for each $x \in X$). But if X is only meta-Lindelöf (i.e., $\{E \in C\colon x \in E\}$ countable) the relation between Σ and S^* can depend very interestingly on whether the continuum hypothesis is true, or whether it is false, but with Martin's axiom true. *Ralph Henstock* (Londonderry)

MR1171794 (93g:26012) 26A39 42A16

Cross, George E. (3-WTRL); **Thomson, Brian S.** (3-SFR)

Symmetric integrals and trigonometric series.

Dissertationes Math. (*Rozprawy Mat.*) **319** (1992), 49 *pp.*

The paper has a good historical introduction, and incorporates part of an unpublished manuscript of J. Mařík that gives estimates and properties on the real line of

$$\mathrm{M}_s^2 F(x,h) = \frac{F(x+h) - F(x-h)}{2h} - \frac{1}{h^2}\int_0^h \{F(x+t) - F(x-t)\}\, dt$$

$(h > 0)$ for integrable functions F. The authors use a second symmetric variation V_s^2 of functions $\xi(x,h)$ of real numbers x and sufficiently small $h>0$ to define a variational integral analogous to the one used in Denjoy-Perron-gauge theory. V_s^2 is proved to be an outer measure, and many special cases of variational equivalence are given. For example, if f is Lebesgue or Denjoy-Perron integrable on $[a,b]$ then f is V_s^2-integrable on $[a,b]$. The converse holds if $f\geq 0$ and is V_s^2-integrable, and if f and $|f|$ are V_s^2-integrable. If f is V_s^2-integrable on $[a,b]$, there is a set B of full measure in (a,b) such that f is V_s^2-integrable on $[c,d]$ for all $c<d$, $c,d\in B$. Sometimes c cannot be a. The second derivative of $x(1-x^2)^{1/2}$ is integrable over $[-1,1]$ but not over $[-1,d]$ $(-1<d<1)$. Additivity over abutting intervals sometimes fails. Close connections with James' P^2-integral and with J. C. Burkill's SCP-integral are given. Mařík's integration by parts formula based on $(GF'-G'F)'=GF''-G''F$ is proved, and then Burkill's integration by parts, the usual form, is given following Sklyarenko. Finally, the results are applied to trigonometric series to give theorems of Mařík, Burkill, W. H. Young, and C. de la Vallée-Poussin. This paper is full of interesting insights into things old and new.

Ralph Henstock (Londonderry)

MR1147382 (92k:26012) 26A45

Thomson, Brian S. (3-SFR)

Symmetric variation.

Real Anal. Exchange **17** (1991/92), *no.* 1, 409–415.

The author deals with functions $f\colon \mathbf{R}\to\mathbf{R}$. Let $E\subset\mathbf{R}$ and let δ be a positive function on E. Then

$$S_\delta(f,E)=\sup\sum_{i=1}^n |f(x_i+h_i)-f(x_i-h_i)|,$$

where the supremum is taken with regard to all sequences $\{[x_i-h_i,x_i+h_i]\}$ of nonoverlapping intervals with centers $x_i\in E$ and with $h_i<\delta(x_i)$. The symmetric variation of f on E, $\mathrm{VS}_f(E)$, is defined by $\mathrm{VS}_f(E)=\inf S_\delta(f,E)$, where the infimum is taken over all positive functions δ on E. VS_f is an outer measure on the real line. The paper presents properties of functions having zero, finite or σ-finite symmetric variation on an arbitrary set. It is shown, e.g., that if VS_f is σ-finite on an interval (a,b) then there is a dense set of subintervals of (a,b) on each of which f has bounded variation.

Pavel Kostyrko (SK-KMSK)

MR1139321 (92h:35042) 35F20 35B45
Bruckner, A. M. (1-UCSB); **Petruska, G.** (H-EOTVO);
Preiss, D. (CS-CHRL); **Thomson, B. S.** (3-SFR)
The equation $u_x u_y = 0$ factors.
Acta Math. Hungar. **57** (1991), *no.* 3-4, 275–278.
The paper deals with the partial differential equation $u_x u_y = 0$ on the plane $\mathbf{R^2}$. The question is whether every solution must be a function of one variable. The authors give an affirmative answer if u is continuous in each variable separately, and if at each point in $\mathbf{R^2}$ at least one of the partial derivatives exists and vanishes.

This extends a result due to W. Jockusch, obtained under the assumption that the function u belongs to $C^1(\mathbf{R^2})$.

Paola Loreti (I-ROME-AS)

MR1078198 (92d:26002) 26-02 26A21 26A24
Thomson, Brian S. (3-SFR)
Derivates of interval functions.
Mem. Amer. Math. Soc. **93** (1991), *no.* 452, vi+96 *pp.*
This paper is an important treatise on the theory of real functions. It contains much more than one might guess after just reading the title. The paper was motivated mainly by some earlier results found by Rogers and Taylor "to determine the nature of continuous non-decreasing functions on $[0,1]$ whose Lebesgue-Stieltjes measures are absolutely continuous with respect to the s-dimensional Hausdorff measure on $[0,1]$. This problem leads naturally to an investigation of derivatives of the form

$$D^s(f,x) = \limsup_{y,z\to x,\, y<x<z} (f(z)-f(y))/(z-y)^s$$

which Besicovitch has called Lipschitz numbers." Generalizing this problem, the author sets out to study the derivatives of an interval-point function (i.p.f.) with respect to another i.p.f. The result is a profound and coherent theory containing most of the classical theorems on real functions, and as an application, also the required results on s-absolute continuity. With this clear and elegant contribution the author combines the merits of an excellent survey article and those of a penetrating research study. It would not be surprising if this material soon appeared as a standard and widely appreciated monograph on the subject. The paper is divided into seven chapters and we review their content in the same order. Section 1 is a very informative introduction.

Section 2: Covering relations. Studying differential ratios, covering relations arise in a natural way: they consist of a bunch of intervals where the given ratio satisfies some prescribed condition. In general, a covering relation (c.r.) β is a set of pairs (I,x), where I is an interval in $\mathbf{R}$ and $x\in I$. Two special cases are applied in this work: β is a full c.r. on a set $E\subset\mathbf{R}$ if for every $x\in E$ there exists $\delta>0$ such that $x\in I^\circ$, $|I|<\delta$ implies $(I,x)\in\beta$; on the other hand, β is fine on E if for every $x\in E$ and $\delta>0$ there exists I such that $x\in I^\circ$, $|I|<\delta$ and $(I,x)\in\beta$ hold.

Section 3: The variation. This is the key chapter to the whole essay; the variations introduced here are the fundamental concepts needed to develop the theory and they "provide the link between derivation properties and measure-theoretic properties of functions".

An interval-point function (i.p.f.) is a real-valued function defined on some covering relation(s). Let h be an i.p.f. defined on β and denote

$$\operatorname{Var}(h,\beta)=\sup\Big\{\sum_{(I,x)\in\pi}|h(I,x)|\colon\ \pi\subset\beta,\ \pi \text{ a packing}\Big\}$$

(packing means a covering relation such that the intervals in different elements cannot overlap). Let E be a set of reals, h an i.p.f.; then the full and fine variational measures are defined and denoted by $h^*(E)=V^*(h,E)=\inf\{\operatorname{Var}(h,\beta);\ \beta \text{ a full covering relation on } E\}$ and $h_*(E)=V_*(h,E)=\inf\{\operatorname{Var}(h,\beta);\ \beta \text{ a fine covering relation on } E\}$, respectively. Having the variational measures at hand, integration is immediate.

Section 4: Derivates. Let h and k be i.p.f. The derivates of h relative to k are $\overline{D}(h,k,x)=\limsup_{\delta\to0+}\{|h(I,x)/k(I,x)|\colon\ |I|<\delta,\ x\in I^\circ\}$ and $\underline{D}(h,k,x)=\liminf_{\delta\to0+}\{|h(I,x)/k(I,x)|\colon |I|<\delta,\ x\in I^\circ\}$. As expected, these derivates are closely related to the variational measures of h and k. In the case of interval functions the derivates belong to the second Baire class (which ensures measurability).

Section 5: Absolute continuity and singularity. Let h and k be i.p.f. and let $E\in\mathbf{R}$. h is absolutely continuous with respect to k, in symbols $h\ll k$, on E if for every $\varepsilon>0$ there exists a δ and a full c.r. β on E such that, whenever $\pi\subset\beta$ is a packing with $\sum_{(I,x)\in\pi}|k(I,x)|<\delta$, then $\sum_{(I,x)\in\pi}|h(I,x)|<\varepsilon$. On the other hand h and k are said to be mutually singular on E ($h\perp k$) if for every $\varepsilon>0$ there exists a full c.r. on E such that every packing $\pi\in\beta$ has a disjoint decomposition $\pi=\pi_1\cup\pi_2$ for which $\sum_{(I,x)\in\pi_1}|h(I,x)|<\varepsilon$ and $\sum_{(I,x)\in\pi_2}|k(I,x)|<\varepsilon$.

Weak absolute continuity and weak singularity are defined analogously by substituting "fine" in place of "full". The relations $\ll$, $\perp$ are preserved by the variational equivalence $\equiv$ and, as one may expect, the variational measures behave similarly to the Lebesgue-Stieltjes measures.

Section 6: Measures. This chapter shows how the theory applies for classical measures (Lebesgue, Lebesgue-Stieltjes, Hausdorff) and we can reasonably well summarize the content by saying that all the classical theorems are readily available if the underlying function (or interval function), which generates the measure, is continuous.

Section 7: Real functions. This chapter is undoubtedly the culmination of the work. It will convince any reader that his efforts to accept another way (actually two more ways) of developing the theory of integration and differentiation are highly rewarded. The main applications are the characterizations of the Hausdorff absolutely continuous and singular functions, but as was the case with the measures, the "standard" real function theory (for continuous functions) quickly follows: Lebesgue's theorem on monotone functions, Lebesgue decomposition, Jordan decomposition, de la Vallée Poussin's theorem, description of singular functions appear (are proved) here with ease and in a wider setting. *G. Petruska* (H-EOTVO-1)

MR1059435 (91e:26008) 26A24 26A15

Thomson, Brian S. (3-SFR)

An analogue of Charzyński's theorem.

Real Anal. Exchange **15** (1989/90), *no.* 2, 743–753.

A result of Z. Charzyński [Fund. Math. **21** (1933), 214–225; Zbl **8**, 344] states that if a function $f\colon \mathbf{R}\to\mathbf{R}$ satisfies the condition that $\limsup_{h\to 0}|[f(x+h)-f(x-h)]/2h|<+\infty$ at every point x, then f is continuous everywhere excepting only at the points of some scattered set. Here, the author establishes what can be viewed as the "even" analogue of this result by showing that if a measurable function $f\colon \mathbf{R}\to\mathbf{R}$ satisfies the condition that $\limsup_{h\to 0}|[f(x+h)+f(x-h)-2f(x)]/2h|<+\infty$ at every point x, then f is continuous everywhere excepting only at the points of some scattered set. (The necessity of the measurability condition is well known.) Actually, the author provides two interesting proofs for this result, one utilizing the Charzyński result, and the other independent of it.

Michael Evans (1-WLEE)

MR1042553 (91b:26010) 26A24

Thomson, Brian S. (3-SFR)

Some symmetric covering lemmas.

Real Anal. Exchange **15** (1989/90), *no.* 1, 346–383.

The purpose of this paper is first to present many of the constructions fundamental to the study of symmetric behavior of functions in the context of covering theorems (or lemmas), and second to apply these covering lemmas. In so doing, the author lays out a convincing case for the simplicity of this covering theory perspective. Many classical theorems are re-proved and several new theorems are presented. The paper begins with an historical view of the study. This is followed by a listing of the covering lemmas. Complete proofs are given but in separate sections later in the article. *Paul D. Humke* (1-OLAF)

MR1013466 (90m:26018) 26A39 42A20

Preiss, D. (CS-CHRL)**; Thomson, B. S.** (3-SFR)

The approximate symmetric integral.

Canad. J. Math. **41** (1989), *no.* 3, 508–555.

The object of this paper is the definition of an integral of the Henstock-Kurzweil type which solves the coefficient problem, i.e. which integrates the sum function $f(x)$ of an everywhere convergent trigonometric series $(*)$ $\frac{1}{2}a_0+\sum_1^\infty(a_n\cos nx+b_n\sin nx)$ and then yields $(*)$ as the Fourier series of f. The various existing integrals with the same capacity are inspired by the fact that, if $G(x)$ is the sum of the twice formally integrated series $(*)$, then $f(x)=\lim_{h\searrow 0}\{G(x+h)-2G(x)+G(x-h)\}/h^2$. The present approach derives from Zygmund's theorem to the effect that the function $F(x)$ defined a.e. as the sum of the once formally integrated series $(*)$ has $f(x)$ as its approximate symmetric derivative.

An interval-point relation β is defined as a collection of pairs (I,x), where I is a nondegenerate closed interval and $x\in I$; and β is called measurable approximate symmetric if there is a measurable set $T\subset \mathbf{R}\times(0,\infty)$ such that $([x-t,x+t],x)\in\beta$ whenever $(x,t)\in T$, and $\limsup_{h\searrow 0}|\{t\in(0,h);\ (x,t)\notin T\}|/h=0$ (where $|\cdot|$ denotes Lebesgue outer measure). The family of these β is denoted by $\mathcal{A}$. Also a finite β is called a packing of $[a,b]$ if I_1,I_2 do not overlap when (I_1,x_1), $(I_2,x_2)\in\beta$ are distinct, and the packing is a partition if, additionally, $[a,b]=\bigcup_{(I,x)\in\beta}I$. The key result is that, if $\{\beta_n\}\subset\mathcal{A}$, there is a set B of full measure such that each β_n contains a partition of every interval $[a,b]$ with endpoints in B.

Given a real-valued function f on $\mathbf{R}$, suppose that there exists an additive interval function F defined for all intervals with endpoints in a set B of full measure so that, for every $\varepsilon > 0$, there exists $\beta \in \mathcal{A}$ which contains a partition of every interval with endpoints in B and is such that, for every packing π in β,

$$\sum_{([y,z],x)\in\pi} |F(y,z) - f((y+z)/2)(y-z)| < \varepsilon.$$

Then f is said to be $\mathcal{A}$-integrable and F is called an indefinite $\mathcal{A}$-integral of f. Under these circumstances, f and F are both measurable, F is everywhere $\mathcal{A}$-continuous, i.e. $\text{ap-lim}_{h\searrow 0} F(x-h, x+h) = 0$, and f is a.e. the $\mathcal{A}$-derivative of F, i.e. $\text{ap-lim}_{h\searrow 0} F(x-h, x+h)/h = f(x)$. The $\mathcal{A}$-integral includes the Denjoy-Perron and so the Lebesgue integral, but a nonnegative $\mathcal{A}$-integrable function is also Lebesgue integrable.

There is a sketch of an $\mathcal{A}$-Perron integral (weaker than the $\mathcal{A}$-integral) defined by use of lower and upper approximate symmetric derivatives. The definition depends on the so far unpublished theorem of C. Freiling and D. Rinne asserting that an additive measurable interval function with an everywhere nonnegative lower approximate symmetric derivative is nonnegative. The detailed construction is carried out in a rather more general setting.

Now let the function f have period 2π, say. If there exists a number c such that, for every $\varepsilon > 0$, there exists $\beta \in \mathcal{A}$, such that, for any partition $x_0 < x_1 < \cdots < x_{n-1} < x_n = x_0 + 2\pi$ with $([x_{i-1}, x_i], (x_{i-1} + x_i)/2) \in \beta$ $(i = 1, \cdots, n)$, $|\sum_1^n f((x_{i-1} + x_i)/2)(x_i - x_{i-1}) - c| < \varepsilon$, then f is said to have the periodic integral c. The $\mathcal{A}$-integral is shown to include the periodic integral.

The proof that the $\mathcal{A}$-integral solves the coefficient problem relies on the integral's ability to integrate approximate symmetric derivatives of measurable functions. Since the periodic integral also has this ability, it provides a particularly simple route to a solution of the coefficient problem.

The last part of this paper is devoted to the relationships between the $\mathcal{A}$-integral and other integrals that solve the coefficient problem, such as the SCP-integral. Simple examples show that neither integral contains the other. Moreover, when a function is integrable in both senses, the corresponding integrals may differ. *H. Burkill* (Sheffield)

MR0988374 (90a:26001) 26A03

Preiss, David (CS-CHRL-MA); **Thomson, Brian S.** (3-SFR)

A symmetric covering theorem.

Real Anal. Exchange **14** (1988/89), *no.* 1, 253–254.

From the text: "By a symmetric full cover of the real line is meant a collection $\mathcal{S}$ of closed intervals with the property that for every real x there is a $\delta(x) > 0$ such that $[x-h, x+h] \in \mathcal{S}$ for every $0 < h < \delta(x)$. It has been shown [Thomson, same journal **6** (1980/81), no. 1, 77–93; MR0606543 (82c:26008)] that for such a collection $\mathcal{S}$ there is a closed denumerable set $C \subset (0,\infty)$ such that $\mathcal{S}$ contains a partition of every interval $[-x, x]$ with $x \notin C$. The simplicity and utility of this result may have led some to overlook an extension that is on occasion more useful.

"Theorem: Let $\mathcal{S}$ be a symmetric full cover on the real line. Then there is a denumerable set N such that $\mathcal{S}$ contains a partition of every interval neither of whose endpoints belongs to N."

MR0873899 (88c:26007) 26A24 26A45

Thomson, Brian S. (3-SFR)

Some remarks on differential equivalence.

Real Anal. Exchange **12** (1986/87), *no.* 1, 294–312.

This interesting paper places some differential concepts of S. Leader [Amer. Math. Monthly **93** (1986), no. 5, 348–356; MR0841112 (87e:26002); Real. Anal. Exchange **12** (1986/87), no. 1, 144–175; MR0873890 (88a:26007)] in the setting of the present author's general differentiation theory [ibid. **8** (1982/83), no. 1, 67–207; ibid. **8** (1982/83), no. 2, 278–442; MR 84i:26008ab; *Real functions*, Lecture Notes in Math., 1170, Springer, Berlin, 1985; MR0818744 (87f:26001)]. Let $h(I,x)$ be an interval point function. It then is said to be (VB)-dampable, according to Leader, if there is a $k(x) > 0$ and a g of bounded variation such that $k(x)h(x,I)$ and $g(I)$ $(= g(\beta) - g(\alpha)$ if $I = [\alpha, \beta])$ are differentially equivalent (roughly speaking, k is the derivative of g with respect to h). One of the main results is: if $f(x)$ is continuous then $f(I)$ is (VB)-dampable if and only if f is VBG^*. *P. S. Bullen* (3-BC)

MR0869719 (88f:26005) 26A24

Bruckner, A. M. (1-UCSB); **Laczkovich, M.** (H-EOTVO); **Petruska, G.** (H-EOTVO); **Thomson, B. S.** (3-SFR)

Porosity and approximate derivatives.

Canad. J. Math. **38** (1986), *no.* 5, 1149–1180.

This is an important and highly technical paper concerned with the relationships between ordinary, approximate, path and sequential derivatives with various porosity conditions. One reason for its importance deals with an early result of A. Ya. Khinchin [Mat. Sb. **31** (1924), 265–285]. This result, by counterexample, is shown to be false (a fact, pointed out by the authors, that may have been known to Khinchin). Since G. H. Sindalovskiĭ has used this result and some of the faulty techniques in one of his papers [Math. USSR-Izv. **2** (1968), no. 5, 943–978; MR0239016 (39 #375)], it appears that some of his results may not be correct.

The authors provide also a correct version of Khinchin's theorem and a necessary and sufficient condition for Sindalovskiĭ's result to hold. The results are rather too technical to restate in this short format, but we include the following from the author's introduction: "A path derivative of a function f is $f'_E(x) = \lim_{y\to x,\, y\in E_x}(f(y) - f(x))/(y-x)$, where at each point x a set E_x is given. A special case of the path derivative is the sequential derivative, $f'_h(x) = \lim_{n\to\infty}(f(x + h_n) - f(x))/h_n$, where $\{h_n\}$ is a fixed sequence of nonzero numbers converging to zero. Two natural questions arise in this setting: (a) what information about f'_E or f'_h on a set A implies that f is differentiable or approximately differentiable a.e. in A and (b) when such derivatives exist on a set A, on which the approximate derivative $f'_{\rm ap}$ also exists, what conditions ensure that $f'_{\rm ap}(x) = f'_E(x)$ or $f'_{\rm ap}(x) = f'_h(x)$ a.e. in A?"

Richard J. O'Malley (1-WIM)

MR0825474 (87g:26007) 26A30 26A24

Thomson, B. S. (3-SFR)

Singular functions.

Rev. Roumaine Math. Pures Appl. **30** (1985), *no.* 9, 793–800.

In this review all functions are real functions defined on a fixed interval $[a,b]$, all sets are subsets of $[a,b]$, and all intervals are closed subintervals of $[a,b]$. A collection $\mathcal{C}$ of intervals is called a full cover of a set E if for each $x\in E$ there is a $\delta(x)>0$ such that the intervals $[x,x+t]$ and $[x-t,x]$ for which $0<t<\delta(x)$ belong to $\mathcal{C}$; $\mathcal{C}$ is called a fine cover of E if for every $x\in E$ and every $\varepsilon>0$ there is a number t for which $0<t<\varepsilon$ so that one of the intervals $[x,x+t]$ and $[x-t,x]$

belongs to C. If h is a real-valued interval function defined at least on C then we write $\mathrm{Var}(h,C)=\sup[\Sigma|h(I)|; I\in\pi]$, where the supremum is taken over all finite subsets $\pi\subset C$ such that I and J do not overlap for distinct I,J in π. We write $\overline{V}(h,E)=\inf[\mathrm{Var}(h,C)\colon C$ is a full cover of $E]$ and $\underline{V}(h,E)=\inf[\mathrm{Var}(h,C)\colon C$ is a fine cover of $E]$.

Let $\mu_f(E)=\overline{V}(f,E)$, where f denotes both the function $x\mapsto f(x)$ and its associated interval function $f\colon[c,d]\to[f(d)-f(c)]$. If $\overline{V}(f,E)=\underline{V}(f,E)$, $\mu_f(E)$ is the variation $V(f,E)$. The functions f and g are mutually singular on E if $V(\sqrt{|f,g|},E)=0$. Necessary and sufficient conditions are given for two functions f and g of finite variation to be mutually singular on E.

Solomon Marcus (Bucharest)

MR0818744 (87f:26001) 26-02

Thomson, Brian S. (3-SFR)

★**Real functions.**

Lecture Notes in Mathematics, 1170.
Springer-Verlag, *Berlin*, 1985. vii+229 *pp.* $14.40.
ISBN 3-540-16058-2

The title of the book is somewhat misleading. This is not a course on real functions but a survey of up-to-date research in this field. It contains a number of topics related to the study of the continuity and differentiation properties of functions of one variable. These are connected with the notions of cluster sets, variation of a function and monotonicity. There are now a large number of papers concerning this subject available in a variety of journals. The author collects and continues certain ideas that arise in those articles. The framework that he uses to formulate new and reformulate old notions of limit, continuity, derivation, etc., is the local system of sets. This is a modification of the concept of filter that is used in topology. By a local system on a set $X\subset\mathbf{R}$ we mean a family $\mathbf{S}$ of sets $S\subset\mathbf{R}$ such that at each point $x\in X$ there is given a nonempty collection $\mathbf{S}(x)$ of sets $S\in\mathbf{S}$ with the following properties: (i) $\{x\}\notin\mathbf{S}(x)$, (ii) if $S\in\mathbf{S}(x)$ then $x\in S$, (iii) if $S_1\in\mathbf{S}(x)$ and $S_2\supset S_1$ then $S_2\in\mathbf{S}(x)$, (iv) if $S\in\mathbf{S}(x)$ and $\delta>0$ then $S\cap(x-\delta,x+\delta)\in\mathbf{S}(x)$. Let $f\colon\mathbf{R}\to\mathbf{R}$. An extended real number c is called an $\mathbf{S}$-limit of f at x if $f^{-1}(U)\setminus\{x\}\in\mathbf{S}(x)$ for every open set U containing c. In a similar manner we define $\mathbf{S}$-derivates and then the $\mathbf{S}$-cluster sets at x as the collection of all $\mathbf{S}$-limits at x. Many examples of local systems are given in the first chapter, among them those connected with approximate continuity, essential continuity, quasicontinuity, negligent limits, etc. From the general definition of the local system there arise the notions of $\mathbf{S}$-

cover of a set, e.g. the **S**-Vitali cover, the **S**-variation of a function and the variation measure.

Chapter two, "Cluster sets", gives the classical material on real cluster sets and develops some abstract presentation of this material. There are three basic types of theorems in this study; these concern a weak form of continuity, asymmetry theorems like Young's Rome theorem, and ambiguity theorems on the set of points at which S_1- and S_2-cluster sets do not have even a single value in common.

A function f is said to be **S**-continuous at x if $f(x)$ belongs to the **S**-cluster set of f at x. This concept permits much more delicate variants of continuity than one-sided continuity, approximate continuity, etc. The properties of the sets of **S**-continuity and **S**-discontinuity points are studied, especially for Baire 1 functions and functions with the Darboux property. The conditions under which **S**-continuity requires continuity in the ordinary sense end Chapter three (entitled "Continuity").

In Chapter four, "Variation of a function", the author explores the notion of bounded **S**-variation. For each function f and each local system **S** there is an outer measure defined which is used to describe various differentiation properties of f. The Denjoy-Luzin theorems on differentiability of a VBG_* function, the de la Vallée-Poussin decomposition theorem, properties of absolutely continuous and of singular functions are generalized and proven.

Chapter five, "Monotonicity", is devoted to conditions under which a function is monotone, first of all to those concerned with the **S**-derivates.

In Chapter six, "Relations among derivatives", the author is concerned with an abstract investigation of classical results of W. H. Young, Denjoy and others. The section contains a variety of related theorems; all of them have as their focus the assertion of connections between two different processes of derivation. These concerns are continued in Chapter seven, "The Denjoy-Young relations". It contains a proof of the Denjoy-Young-Saks theorem and discusses several variants of this theorem obtained by a number of investigators, among them the approximate version of the theorem. Some results are expressed in terms of "porosity limits". The exceptional sets recognized in classical theorems as zero sets or sets of the first category appear as σ-porous sets.

The book concludes with a useful appendix on porous sets. It may be used as an introduction to the concept and as a point of reference. Connections between the thinness of a set and its thinness given in the sense of category, measure and Hausdorff measure are studied.

Many nontrivial examples of porous, nonporous and σ-porous sets are given.

The book gives a fairly complete account of results obtained up to very recently. This includes works by A. M. Bruckner, K. M. Garg, P. D. Humke, J. M. Jędrzejewski, M. Laczkovich, R. J. O'Malley, T. Świątkowski, W. Wilczyński, L. Zajiček and others. A part of the material is taken from the research of the author. This is a treatise gathering new results and setting them into the context of the current state of the theory of real functions. It is a thoroughly professional contribution to the literature and the author is to be commended for producing it. *J. S. Lipiński* (PL-GDAN)

MR0807990 (87h:26002) 26A15

Thomson, B. S. (3-SFR)

On the level set structure of a continuous function.

Classical real analysis (*Madison*, *Wis.*, 1982), 187–190, *Contemp. Math.*, 42, *Amer. Math. Soc.*, *Providence*, *RI*, 1985.

Given $A \subset \mathbf{R}$, $x \in A$ and $h > 0$, let $\lambda(A, x, h)$ denote the length of the largest open interval in $(x, x+h)$ that lies in the complement of A. The set A is said to be strongly porous on the right at x if $\limsup_{h\to 0+} \lambda(A, x, h)/h = 1$. Strong porosity on the left is defined similarly.

The author obtains two theorems dealing with the thinness of level sets of continuous functions. (1) If a continuous function $f\colon \mathbf{R} \to \mathbf{R}$ is nowhere constant, then there is a residual set E in $\mathbf{R}$ such that for each $x \in E$ the level set $f^{-1}(f(x))$ is strongly porous on both sides at x. In the case when f is nowhere monotone, it was proved earlier by the reviewer [Fund. Math. **52** (1963), 59–68; MR0143855 (26 #1405)] that each $x \in E$ is a bilateral limit point of $f^{-1}(f(x))$. Now let Φ denote the Banach space of all continuous 2π-periodic functions $f\colon \mathbf{R} \to \mathbf{R}$ furnished with the sup norm. (2) There is a residual set C in Φ such that, for each $f \in C$, every level set of f is strongly porous on both sides at each of its points. Further properties of level sets of $f \in C$ were obtained earlier by A. M. Bruckner and the reviewer [Trans. Amer. Math. Soc. **232** (1977), 307–321; MR0476939 (57 #16487)].

{For the entire collection see MR0807970 (86f:00013)}

K. M. Garg (3-AB)

MR0807980 (86m:26004) 26A21 54H05

Humke, P. D. (1-OLAF); **Thomson, B. S.** (3-SFR)

A porosity characterization of symmetric perfect sets.

Classical real analysis (*Madison*, *Wis.*, 1982), 81–85, *Contemp. Math.*, 42, *Amer. Math. Soc.*, *Providence*, *RI*, 1985.

Starting with a sequence of numbers e_n between 0 and $\frac{1}{2}$ and $a_n = 1 - 2e_n$, an open interval of length $(b-a)\cdot a_1$ is removed from the center of the interval $[a,b]$. Open intervals of length $L_1 a_2$ are removed from the center of the remaining intervals whose length is L_1. Repeating this process for each natural number n leaves behind what the authors refer to as a symmetric perfect set. They show that this set is non-σ-porous if and only if e_n approaches $\frac{1}{2}$ and note that the set has positive measure if and only if $2^n \prod e_n \neq 0$. Alternatively, they note that the set is non-σ-porous if and only if a_n approaches 0 and has positive measure if and only if $\sum a_n < \infty$. Every σ-porous set is of first category and of measure 0. This paper provides a plethora of examples of sets of measure 0 and first category which are not σ-porous.

{For the entire collection see MR0807970 (86f:00013)}

James Foran (1-MOKC)

MR0807973 (87b:26008) 26A24

Bruckner, A. M. (1-UCSB); **O'Malley, R. J.** (1-WIM); **Thomson, B. S.** (3-SFR)

Path derivatives: a unified view of certain generalized derivatives.

Classical real analysis (*Madison*, *Wis.*, 1982), 23–27, *Contemp. Math.*, 42, *Amer. Math. Soc.*, *Providence*, *RI*, 1985.

This article is a summary of the excellent paper of the same title which appeared earlier [Trans. Amer. Math. Soc. **283** (1984), no. 1, 97–125; MR0735410 (86d:26007)].

{For the entire collection see MR0807970 (86f:00013)}

Michael Evans (1-WLEE)

MR0735410 (86d:26007) 26A24

Bruckner, A. M. (1-UCSB); **O'Malley, R. J.** (1-WIM);
Thomson, B. S. (3-SFR)

Path derivatives: a unified view of certain generalized derivatives.

Trans. Amer. Math. Soc. **283** (1984), *no.* 1, 97–125.

In this paper the authors introduce the concept of the path derivative and present the fundamental properties of this important new tool in differentiation theory. The first three sections provide introduction, preliminaries and some basic facts.

A subset $E_x \subset \mathbf{R}$ of the real line $\mathbf{R}$ is called a path leading to x if $x \in E_x$ and x is an accumulation point of E_x. Given a system of paths $E = \{E_x : x \in \mathbf{R}\}$ we can define the path derivative of a real function F with respect to the system E by $F'_E = \lim_{y \to x, y \in E_x} (f(y) - f(x))/(y - x)$. The notions "$E$-differentiable", "$E$-primitive", "$E$-derivative", "extreme E-derivates" are now obvious. Many of the well-known generalized derivatives can be expressed as a path derivative with respect to a suitably chosen system of paths. The authors recall Jarnik's theorem (Theorem 3.1): There exists a universal continuous function F such that for any given function f on $\mathbf{R}$ a system of paths E can be found with $F'_E = f$.

This clearly indicates that being the E-derivative of F implies nothing more on f than the trivial fact that $f(x)$ is a derived number of F at every x. This motivates one to introduce more restrictive conditions on E. E is said to satisfy the intersection condition (I.C.) if a function $\delta(x) > 0$ exists such that whenever $0 < y - x < \min(\delta(x), \delta(y))$ then $E_x \cap E_y \cap [x, y] \neq \varnothing$. The internal intersection condition (I.I.C.) means $E_x \cap E_y \cap (x, y) \neq \varnothing$ and the external intersection condition with parameter m (E.I.C.[m]) means $E_x \cap E_y \cap (y, y + m(y - x)) \neq \varnothing$, $E_x \cap E_y \cap (x - m(y - x), x) \neq \varnothing$. The case $m = 1$ is simply labelled as E.I.C.

Also, bilateral paths and nonporous paths are often assumed. For example, an approximately differentiable function F admits a choice of E such that each path in E has density 1 at x and $F'_{\text{app}} = F'_E$. In this case E_x is bilateral, nonporous at x and satisfies any of the intersection conditions defined above. Moreover, the majority of the properties of the approximate derivative depend upon these conditions on E and hence they are immediate for generalization. As for the selective derivative (introduced by O'Malley [Acta Math. Acad. Sci. Hungar. **29** (1977), no. 1-2, 77–97; MR0437690 (55 #10614)]), it is shown that the E-derivative with respect to a bilateral I.I.C. system is also a selective derivative relative to a suitable selection. However, it is an open problem whether every selective derivative can be represented as a path derivative relative to such a system.

In Section 4 the results are formulated in terms of the extreme E-derivates. As generalizations of the results of G. C. Young and W. H. Young, it is shown that $\bar{F}'_E(x) < \underline{F}'_{E^*}(x)$ can only hold on a countable set if the systems E and E^* satisfy I.C.; and also, if E satisfies I.C., then the extreme E-derived numbers agree with the corresponding ordinary extreme Dini derivates everywhere off a set of first category. Denjoy's classical theorem directly generalizes as follows: If E satisfies I.C. and everywhere on a set X one of the extreme derivates $\underline{F}'_E$ or $\bar{F}'_E$ is finite, then F is of generalized bounded variation (VBG) on X; if both of these are finite, then F is generalized absolutely continuous (ACG) on X. Next, monotonicity theorems are obtained; for instance, $\underline{F}'_E \geq 0$ a.e. and $\underline{F}'_E > -\infty$ $(x \in [a,b])$ imply that F is increasing (again, I.C. and bilateral paths are assumed).

Properties of E-primitives are discussed in Section 5. The next result has been obtained earlier for several generalized derivatives and it clearly shows the unifying strength of the E-derivative. Let F be E-differentiable relative to a system E satisfying any of the intersection conditions. Then, F is absolutely continuous on the components of an everywhere dense open set. In particular, F is differentiable a.e. on this set and approximately differentiable a.e. on $\mathbf{R}$. This result is sharpened in a later section under more restrictive conditions: If E has nonporous paths and satisfies both I.C. and E.I.C., then an E-differentiable function is differentiable on an everywhere dense open set. Section 6 deals with the path derivatives. F'_E belongs to the first class of Baire, if E has E.I.C.$[m]$. The Darboux property holds for F'_E if it is Baire 1 and E is bilateral with I.C. An E-derivative has the Denjoy property, if the paths of E are bilateral and E satisfies both I.C. and E.I.C.$[m]$.

It is also shown that the porosity of the associated sets of F'_E is directly connected to that of the paths in E. As regards Zahorski's $\mathfrak{M}_3$ property, the following result is obtained: Let E be a system of nonporous paths satisfying I.C. and E.I.C. Then every E-derivative has the $\mathfrak{M}_3$ property.

Section 7 gives criteria for the differentiability of E-primitives and miscellaneous results are collected in Section 8. The important O'Malley-Weil "$-M, M$" theorem on the approximate derivative is generalized as follows. Let E be a nonporous system of paths satisfying I.C. If F is E-differentiable and F'_E is Baire 1, then whenever F'_E attains the values M and $-M$ on an interval I_0, there is a subinterval $I \subset I_0$ on which F is differentiable and F' attains both values M and

$-M$.

Many interesting applications are mentioned. This well-written paper is an extremely useful contribution to differentiation theory. It clarifies the essential conditions on which the much-investigated properties of generalized derivatives really depend. It will serve researchers in this area as a map serves explorers of the jungle.

{See also the following review.} *G. Petruska* (H-EOTVO-1)

MR0766077 (86b:26006) 26A24 26A27

Bruckner, A. M. (1-UCSB)**; Thomson, B. S.** (3-SFR)

Porosity estimates for the Dini derivatives.

Real Anal. Exchange **9** (1983/84), *no.* 2, 508–538.

For any function $f\colon \mathbf{R}\to\mathbf{R}$, let $A(f)$ denote the set of points at which the upper right Dini derivate differs from the upper left Dini derivate. The classical paper of W. H. Young [Proc. London Math. Soc. (2) 6 (1908), 298–320; Jbuch 39, 467] states that if f is continuous, then $A(f)$ is a first category set. P. D. Humke and the reviewer [Proc. Amer. Math. Soc. 79 (1980), no. 4, 609–613; MR0572313 (81h:26002)] have shown that if f satisfies a Lipschitz condition, then $A(f)$ is a σ-porous set. The present paper uses the notions of porosity and (Ψ)-porosity to form a framework to unify and extend these and related theorems. One of the main results is the following. Let Ψ be a continuous increasing function on $[0,+\infty)$ for which $\Psi(0)=0$ and $\Psi'_+(0)=+\infty$, and let f be a continuous function satisfying $|f(x)-f(y)|\le\Psi(|x-y|)$ for $|x-y|\le 1$. Then at every point x with the exception of a set that is σ-(Ψ)-porous, and for every $0\le p<1$, the left and right upper porosity Dini derivates of index p of f at x agree with the ordinary upper Dini derivate of f at x. (The various definitions are too technical to include here, but to anyone familiar with porosity, they are the natural ones.) Several other interesting theorems and examples are presented. *Michael Evans* (1-WLEE)

MR0686501 (84c:26012) 26A24
Thomson, B. S.
Some theorems for extreme derivates.
J. London Math. Soc. (2) **27** (1983), *no.* 1, 43–50.
In recent years, two classical theorems the Dini derivates of functions have been extended to certain generalized derivatives. One of these theorems asserts that for a continuous function, the two unilateral upper derivates agree except on a first category set. The other theorem asserts that for arbitrary functions, a lower derivate can exceed the opposed upper derivate on at most a denumerable set. The present author obtains versions of these theorems in the setting of path derivatives, a rather general setting which embraces many of the known notions of generalized differentiation. He shows that an analogue of the first theorem is valid whenever the system of paths satisfies a rather weak porosity condition. An analogue of the second theorem is available whenever the system of paths satisfies a certain intersection condition. In particular, the theorem holds in the setting of approximate differentiation. The article contains all the necessary technical language, as well as some historical discussion of settings in which the results are known to be valid or invalid.

A. M. Bruckner (Santa Barbara, Calif.)

MR694 507 (84i:26008a) 26A24 26A39 28A15
Thomson, B. S.
Derivation bases on the real line. I.
Real Anal. Exchange **8** (1982/83), *no.* 1, 67–207.

MR0700194 (84i:26008b) 26A24 26A39 28A15
Thomson, B. S. (3-SFR)
Derivation bases on the real line. II.
Real Anal. Exchange **8** (1982/83), *no.* 2, 278–442.
Differentiation is a local process, but the study of its properties needs global constructions such as divisions or partitions of sets. The author uses derivation bases, i.e. nonempty collections of families of interval-point pairs, instead of the reviewer's division space ideas, to unify the treatment of derivatives, Kurzweil's special and the reviewer's more general types of generalized Riemann integration, and the reviewer's variation. de Guzmán has studied the differentiation of absolute integrals while the present author studies differentiation and absolute and nonabsolute integration, but only on the real line, ignoring derivatives and integrals defined by convergence-factors, higher order derivatives,

and vector-valued functions. Even with these limitations a large number of derivation processes are considered of the form $\underline{\mathrm{GD}}\ h_g(x) = \liminf h(I)/g(I)$, $\overline{\mathrm{GD}} h_g(x) = \limsup h(I)/g(I)$, where the interval I tends to the point x in a sense involving derivation bases. Usually $g(I) = |I|$, the length of I. Chapter 1, p. 70, introduces the theory with a wealth of special examples, Chapter 2, p. 91, deals with derivation bases, and Chapter 3, p. 137, discusses the variation. This is an outer measure when the derivation base has σ-local character (or is decomposable). Further, Baire 1 properties and various decompositions are given. The whole theory prepares the way for the second paper and 118 references are given.

{Reviewer's remarks: In two places, p. 88 line 4 and p. 118 line 2, the square brackets need to contain the respective numbers 110 and 58. Other trivial typing errors can easily be corrected.}

Part II has Chapter 4, p. 280, on the derivation theory and Chapter 5, p. 362, on the integration theory. The paper has a slight refinement using simple systems of sets and constructing derivation bases from them (pp. 280–283) and proceeds to connections between the two. It is shown that in various generalized senses, a zero derivative implies a constant primitive, a bounded derivative implies bounded variation, and an exact finite derivative implies an ACG primitive, and the fundamental theorem of calculus holds in both directions in that the derivative of the integral is the integrand, while the integral of the derivative recovers the original function to within a constant, even in the general senses. Monotonicity follows from the nonnegativeness of the lower derivate in a variety of senses given in Section 6, Chapter 4, while relations between derivates using various bases are discussed in Section 7, including Denjoy, Beppo Levi, Young, and Khinchin relations. Sections 8, 9 discuss properties of exact derivatives. Chapter 5 gives a discussion of the many varieties of generalized Riemann integral, with Cauchy-type existence of limits of sums, and proceeds to the variational, Ward, and Perron integrals, with properties including limits under the integral sign (except dominated convergence), the Cauchy and Harnack extensions, and integration by parts. In an appendix the author explains the notion of set porosity that has connections with differentiation and that began with Denjoy. There are 338 references.

{Reviewer's remark: Concerning p. 279 lines 18, 19, the reviewer cannot claim H. W. Pu as his student, nor P. McGill, except as a valued colleague for some years and a student of J. J. McGrotty who was the reviewer's first research student. On p. 261 lines 10, 15, for D_s read $\overline{D}_s$, and for F' read $\overline{F}'$. Any further corrections are easy to make. The author has given a magnificent theory in the two papers, and the reviewer looks forward to reading the proposed book.}

Ralph Henstock (Coleraine)

MR0627688 (83b:26014) 26A48 28A15

Thomson, B. S.

Monotonicity theorems.

Proc. Amer. Math. Soc. **83** (1981), *no.* 3, 547–552.

The author introduces the notion of an abstract derivation basis and interprets several monotonicity theorems in terms of the "geometry" of the derivation basis. This work can be considered a part of an extensive plan of the author to unify a number of concepts in real analysis through the use of differentiation bases. The details of the approach are too technical to be reproduced here but the interested reader is referred to the expository article [MR0623052 (83b:26015)below].

The reviewer echoes the comments of A. M. Bruckner in his review of another article by the author [J. London Math. Soc. (2) **22** (1980), no. 3, 473–485; MR0596326 (81m:26004)] in connection with this paper.

Richard J. O'Malley (Milwaukee, Wis.)

MR0632744 (83a:26017) 26A45 28A12

Thomson, B. S.

Outer measures and total variation.

Canad. Math. Bull. **24** (1981), *no.* 3, 341–345.

The author gives some theorems on the outer measures ψ_f and ψ^f that have been introduced in a companion paper [83a:26016 above] and are connected with the Henstock total variation measures. For instance, if $f\colon \mathbf{R}\to\mathbf{R}$ is continuous and has locally bounded variation then the measures ψ_f and ψ^f are identical. For every function $f\in C[a,b]$ excepting a subset of the first category in that space, the measure ψ_f vanishes and the measure ψ^f is non-σ-finite on each subinterval of $[a,b]$.

J. S. Lipiński (Gdańsk)

MR0632743 (83a:26016) 26A45 26A24 28A12

Thomson, B. S.

On the total variation of a function.

Canad. Math. Bull. **24** (1981), *no.* 3, 331–340.

The problem in general is to provide a construction of a measure from a completely arbitrary function in such a way that the values of this measure provide information about the total variation of the function over sets of real numbers and from which useful inferences can be drawn. In the study of the derivation properties of the function there is a natural method of constructing some useful variation measures. The author uses such ideas to give general variation measures that answer some problems in derivation theory. Let $\mathbf{R}$ be the set of all real numbers and J the collection of all closed intervals. If $S \subset J \times \mathbf{R}$ and $h: J \times \mathbf{R} \to \mathbf{R}$ then put $V(h,S) = \sup \sum |h(I_i, x_i)|$, where $\{(I_i, x_i)\}$ is a sequence of interval-point pairs from S with pairwise nonoverlapping elements I_i. Put $V(h,\varnothing) = 0$ and $V(h,\mathfrak{A}) = \inf\{V(h,S): S \in \mathfrak{A}\}$, where $\mathfrak{A}$ is any nonempty family of subsets of $J \times \mathbf{R}$. Let $S[X] = \{(I,x) \in S: x \in X\}$ and $\mathfrak{A}[X] = \{S(X): S \in \mathfrak{A}\}$. Then the author defines the outer measures $\psi^f(X)$ and $\psi_f(X)$ as $V(h,\mathfrak{A}[X])$, where h is the function $h([a,b],x) = f(b) - f(a)$ and $\mathfrak{A}$ can be chosen separately to yield either measure.

The classical Peano-Jordan measures, the Lebesgue-Stieltjes measures, and the Henstock variation and inner variation appear to be particular cases of this general notion. *J. S. Lipiński* (Gdańsk)

MR0623052 (83b:26015) 26A48 28A15

Thomson, B. S.

Monotonicity theorems.

Real Anal. Exchange **6** (1980/81), *no.* 2, 209–234.

This is a well-written readable exposition of the author's plan to apply abstract differentiation theory to monotonicity theorems. (Actually the concepts developed here have a broader application but only the monotonicity aspects are discussed here.) A superficial sketch of the contents follows: (1) The concepts of an abstract derivation basis B and the extreme derivates of a function f relative to B are defined. (2) As illustration and motivation, examples of various bases which yield traditional ideas of derivative or derivate are given. (3) Five properties are described: filtering down, pointwise character, finer than the topology, partitioning and Young decomposition. In each instance, it is clearly explained what bearing each property has to the general problem. (4) For a derivation basis B having these properties, illustrations are given of monotonicity theorems which follow from the behavior of the extreme B derivatives. (5) Total variation measures are described and their properties enumerated. It is shown that in this study of monotonicity their use leads to the concept of H-completeness. (6) Finally H-completeness, in conjunction with the other properties, is shown to yield several monotonicity results.

Richard J. O'Malley (Milwaukee, Wis.)

MR0606543 (82c:26008) 26A24

Thomson, B. S.

On full covering properties.

Real Anal. Exchange **6** (1980/81), *no.* 1, 77–93.

The purpose of this paper is to promote the notion of full covers and their associated full covering lemmas in contrast to Vitali covers and their associated covering lemmas. A collection of closed subintervals of $[a,b]$ is called an ordinary full cover of $[a,b]$ if to each x in $[a,b]$ there is a $\delta(x) > 0$ such that if I is a closed interval with $x \in I$ and $|I| < \delta(x)$, then I is in the collection. The corresponding covering lemma is that an ordinary full covering of $[a,b]$ contains a partition of every closed subinterval. With the aid of the lemma two standard monotonicity theorems are proved. If the lower derivate $\underline{f}'(x) \geq 0$ on $[a,b]$, then f is nondecreasing. If $f'(x) \geq 0$ a.e. on $[a,b]$ and if $f'(x) > -\infty$ on $[a,b]$, then f is nondecreasing. Next, a collection of closed intervals is called an approximate full cover of $[a,b]$ if to each x in $[a,b]$ there is a measurable set A_x having density 1 at x such that every closed interval I with x in I and endpoints in A_x belongs to the collection. The covering lemma in this case is obtained from the first one by replacing "ordinary" by "approximate" and the monotonicity theorems proved are those constructed by replacing $\underline{f}'$ by $\underline{f}'_{\rm ap}$ in the above monotonicity theorems. Other types of full covers are defended, discussed and applied.

Clifford E. Weil (E. Lansing, Mich.)

MR0596326 (81m:26004) 26A24 26A45

Thomson, B. S.

On the derived numbers of VBG_* **functions.**

J. London Math. Soc. (2) **22** (1980), *no.* 3, 473–485.

The author presents a perspective from which one can view a number of classical results related to the differentiation theory of real functions. He begins by associating with each real-valued function f, defined on an interval I, a pair of measures Δf^* and Δf_*. These measures describe the total variation of f and carry information about the derived numbers of f. He first proves Theorem 3.1: If f is continuous on $[a,b]$ and VBG* on a set $X \subset [a,b]$, then Δf^* is σ-finite on X and $\Delta f^*(Y) = \Delta f_*(Y)$ for every $Y \subset X$. He then proceeds to use these measures to obtain a number of results related to absolute continuity, existence of the derivative, vanishing of the derivative, versions of the fundamental theorem of calculus and monotonicity theorems. Most of these are rather deep classical results (or variants of such results), yet they follow relatively easily by his methods. In particular, Theorem 3.1 stated above and its consequences play an important role in the proofs.

{Reviewer's remarks: This paper is apparently part of the author's program to unify a number of concepts in real analysis through the use of differentiation bases. The reviewer has seen several preprints on the subject and believes the program shows promise of clarifying the ways in which a number of classical concepts interrelate and of pointing to further directions of research. The term "differentiation basis" does not appear explicitly in the paper under review. Instead, the author uses the terms "full cover" and "fine cover".}

A. M. Bruckner (Santa Barbara, Calif.)

MR0511583 (80m:28004) 28A15

Thomson, B. S.

On weak Vitali covering properties.

Canad. Math. Bull. **21** (1978), *no.* 3, 339–345.

Using the original method of R. de Possel [J. Math. Pures Appl. **15** (1936), 391–409; Zbl **15**, 205], the author gives covering conditions that are necessary and sufficient for a basis to differentiate a given class of integrals.

D. Preiss (Prague)

MR0476979 (57 16524) 28A15

Thomson, B. S.

Measures generated by a differentiation basis.

Bull. London Math. Soc. **9** (1977), *no.* 3, 279–282.

The author shows how a new point of view can be taken in the theory of differentiation of set functions. Instead of considering, as usual, differentiation bases with special properties which guarantee the existence of nice differentiation properties, one can start with an abstract differentiation basis $\mathcal{F}$ and an outer measure μ. One then defines certain auxiliary measures μ_λ and μ_ν, and under rather mild conditions one obtains density theorems for μ with respect to $\mathcal{F}$ at μ_λ- and μ_ν-almost every point. *M. de Guzman* (Madrid)

MR0463380 (57 3332) 26A42 28A25

Thomson, B. S.

A characterization of the Lebesgue integral.

Canad. Math. Bull. **20** (1977), *no.* 3, 353–357.

After J. Kurzweil [Czechoslovak Math. J. **7 (82)** (1957), 418–449; MR0111875 (22 #2735)] and the reviewer [Proc. London Math. Soc. (3) **10** (1960), 281–303; MR0121460 (22 #12198); ibid. (3) **11** (1961), 402–418; MR0132147 (24 #A1994)] had independently characterized Denjoy-Perron integration using Riemann sums but a limit different from the norm-limit, E. J. McShane [*A Riemann-type integral that includes Lebesgue-Stieltjes, Bochner and stochastic integrals*, Mem. Amer. Math. Soc., No. 88, Amer. Math. Soc., Providence, R.I., 1969; MR0265527 (42 #436)] characterized the case in which both f and $|f|$ are integrable, so that f is Lebesgue integrable, by altering the limit slightly. McShane showed that this case is equivalent to a certain absolute integrability, but did not prove directly the equivalence to Lebesgue integration. The present author gives an interesting direct proof of the equivalence of this absolute integrability and Lebesgue integrability using Luzin's criterion and the a.e. approximate continuity criterion for measurability. *Ralph Henstock* (Coleraine)

MR0422550 (54 10537) 28A10

Thomson, B. S.

Construction of measures in metric spaces.

J. London Math. Soc. (2) **14** (1976), *no.* 1, 21–24.

L'auteur appelle prémesure une fonction τ d'ensemble définie sur une class $\mathbf{C}$ de sous-ensembles d'un ensemble Ω si $\varnothing \in \mathbf{C}$, $0 \le \tau C \le +\infty$ pour $C \in \mathbf{C}$, et $\tau\varnothing = 0$. Il appelle mesure une prémesure μ définie sur tous les sous-ensembles de Ω et telle que $A \subset \bigcup_{i=1}^{\infty} B_i$ implique que $\mu A \le \sum_{i=1}^{\infty} \mu B_i$. Il appelle mesure métrique une mesure μ définie sur un espace métrique Ω ayant la métrique ρ et telle que $\inf\{\rho(x,y)\colon x \in A, y \in B\} > 0$ implique que $\mu(A \cup B) = \mu A + \mu B$.

Il indique diverses méthodes de construction de mesures métriques à partir de prémesures, et compare ces mesures métriques entre elles. Il considère le cas où la prémesure dont on part est la restriction d'une certaine mesure métrique. Notons que ce que l'auteur appelle mesure coïncide avec ce que le rapporteur avait appelé fonction de Carathéodory ou fonction carathéodoryenne [A. Appert, Rend. Circ. Mat. Palermo. **12** (1963), 330–346; MR0169965 (30 #208)].

REVISED (1978)

A. Appert

MR0364586 (51 840) 28A15

Thomson, B. S.

Covering systems and derivates in Henstock divison spaces. II.

J. London Math. Soc. (2) **10** (1975), 125–128.

In Part I [same J. (2) **4** (1971), 103–108; MR0294594 (45 #3664)] the author used the idea of inner variation due to R. Henstock [*Theory of integration*, Butterworths, London, 1963; MR0158047 (28 #1274)] to develop a theory of derivates in a general setting. He now obtains a Vitali property for the Henstock division systems by imposing an appropriate halo assumption.

S. J. Taylor

MR0304607 (46 3742) 28A45

Thomson, B. S.

On McShane's vector-valued integral.

Duke Math. J. **39** (1972), 511–519.

In this paper, E. J. McShane's concept of "absolute integrability" [*A Riemann-type integral that includes Lebesgue-Stieltjes, Bochner and stochastic integrals*, Mem. Amer. Math. Soc., No. 88, Amer. Math. Soc., Providence, R.I., 1969; MR0265527 (42 #436)] of a function on $I = Q \times T$ (Q being a ring of subsets of a set T) taking values in a normed linear space is studied in a context derived mainly from Henstock's version of a generalization of the classical Riemann integral [see R. Henstock, Proc. London Math. Soc. (3) **19** (1969), 509–536; MR0251189 (40 #4420)]. Among other things, the author shows how an additive absolutely integrable function on I taking values in a Banach space in a decomposable partitioning system can give rise to a countably additive vector measure on a semitribe of measurable subsets of T. *A. Mukherjea*

MR0304601 (46 3736) 28A25

Thomson, B. S.

A theory of integration.

Duke Math. J. **39** (1972), 503–509.

The author presents an account of the ideas fundamental to a theory of integration investigated by R. Henstock [*Theory of integration*, Butterworths, London, 1963; MR0158047 (28 #1274)] and E. J. McShane [*A Riemann-type integral that includes Lebesgue-Stieltjes, Bochner and stochastic integrals*, Mem. Amer. Math. Soc., No. 88, Amer. Math. Soc., Providence, R.I., 1969; MR0265527 (42 #436)]. The author (like Henstock) constructs an outer measure in a decomposable division system and obtains a monotone convergence property for measures satisfying a regularity assumption. Then he constructs an upper integral that gives rise to a theory of integral in a standard manner. The paper concludes with a brief application of the theory to integration in locally compact spaces. *A. Mukherjea*

MR0299751 (45 8799) 28A45 26A39

Thomson, B. S.

On the Henstock strong variational integral.

Canad. Math. Bull. **14** (1971), 87–99.

The main purpose of the author is to construct a theory of the Henstock strong variational integral [R. Henstock, Proc. London Math. Soc. (3) **19** (1969), 509–536; MR0251189 (40 #4420)] which, while slightly less general, is simpler and sufficient for most applications. Let T denote a set, J a collection of sets of the form (I,x) with $I\subset T$ and $x\in T$, $\mathcal{D}$ is said to be a division if $\mathcal{D}$ is a finite subset of J and the I for which $(I,x)\in\mathcal{D}$ are disjoint sets. Let $\sigma(\mathcal{D})=\bigcup\{I\colon(I,x)\in\mathcal{D}\}$. A set in $\sigma(\mathcal{D})$ is called an elementary set. If $X\subset T$ and $S\subset J$ then $S(X)=\{(I,x)\in S\colon I\subset X\}$ and $S[X]=\{(I,x)\in S\colon x\in X\}$. $(T,\mathfrak{A},J)$ is called a division space if $\mathfrak{A}$ is a collection of subsets of J such that every $S\in\mathfrak{A}$ contains a division of each elementary set; moreover if S_1 and S_2 are in $\mathfrak{A}$, there exists some $S\in\mathfrak{A}$ such that $S\subset S_1\cap S_2$. A division space $(T,\mathfrak{A},J)$ is decomposable if for every sequence of disjoint subsets X_k of T and each $S_k\in\mathfrak{A}$, there exists an $S\in\mathfrak{A}$ such that $S[X_k]\subset S_k[X_k]$. The property of being decomposable plays a role similar to countable additivity in measure spaces. An example of a division space is provided by $T=R^n$, $J=\{[a,b),\chi\}$ where $x\in[a,b]$ and $\mathfrak{A}=\{S_\delta\colon\delta>0\}$, S_δ being the set of all $\{[a,b),\chi\}$ with $x\in[a,b]$ and $[a,b]$ being a subset of the sphere of center x and radius δ. The following functions play an important role. Let h be a function defined on J with values in a normed linear space. For $S\subset J$, set $V(h,S)=\sup(\mathcal{D})\sum\|h(I,x)\|$, where the sup is taken over all $\mathcal{D}\subset S$ and where $(\mathcal{D})\sum$ denotes summation over all $(I,x)\in\mathcal{D}$, an empty sum being zero. Let $\mathfrak{A}(X)=\{S(X)\colon S\in\mathfrak{A}\}$, $\mathfrak{A}[X]=\{S[X]\colon S\in\mathfrak{A}\}$. $V(h,\mathfrak{A}')=\inf V(h,S)$ where the inf is taken over $S\in\mathfrak{A}'$ and $\mathfrak{A}'$ is a family of subsets of J, $V(h,S,X)=V(h,S[X])$, $h^*(X)=V(h,\mathfrak{A}[X])$. The first theorem shows that h^* (defined on subsets of T) behaves like an outer measure and that the countable subadditivity holds if $(T,\mathfrak{A},J)$ is decomposable. A function H defined on elementary sets of $(T,\mathfrak{A},J)$ with values in a normed linear space is said to be additive if $H(E)=(\mathcal{D})\sum H(I)$, where $I\in\mathcal{D}$ and $\mathcal{D}$ is a division of E. H then can be defined on J by $H(I,x)=H(I)$.

Finally h is said to be integrable on $(T,\mathfrak{A},J)$ if there exists an additive H such that $V(H-h,\mathfrak{A})=0$; one writes $\int h=H$, $\int_E h=H(E)$. If $V(h,\mathfrak{A})<+\infty$ then h is said to be summable. An important special case is provided by the following: If E,F,G are Banach spaces and u is a bilinear function from $E\times F$ into G with $|u(x,y)|\le|x||y|$, let m map J into F and let f be a function from T into E;

let $fm(I,x) = u(f(x), m(I,x)) \in G$. The space $\mathfrak{L}_E(m)$ is defined to consist of all functions from T into E which are summable with $\|f\|_m = V(|f||m|, \mathfrak{A}) < \infty$. $\mathfrak{L}_E(m)$ has properties similar to m-summable E valued functions, in particular $(S, \mathfrak{A}, J)$ being decomposable implies that $\mathfrak{L}_E(m)$ is complete in the semi-norm $\|(\cdot)\|m$. As an application of the general theory the case of integration on locally compact spaces is considered and a representation theorem for dominated linear maps on E-valued functions with compact support is obtained.

A. de Korvin