MR2143414 (2006c:26016) 26A39 28A75 49J45

De Pauw, Thierry

Autour du théorème de la divergence. (French. English, French summary) [On the divergence theorem]

Autour du centenaire Lebesgue, 85–121, *Panor. Synthèses*, 18, *Soc. Math. France*, *Paris*, 2004.

This survey appears in a volume produced for the occasion of the centennial of Lebesgue's publication in 1901 of his famous note introducing his integral. The author's contribution was invited to include some of the theory of nonabsolute integration. He chose to focus on the divergence theorem, beginning with the one-dimensional case, i.e., the fundamental theorem of the calculus.

The first part of the paper is a fairly complete survey, with an adequate level of motivation and detail, of the Henstock-Kurzweil integral. While there are a great many extensive accounts of this material in English, this is perhaps only the second in French (after [J. L. Mawhin, *Analyse*, Second edition, De Boeck Univ., Brussels, 1997; MR1438786 (97k:00001)]). This one-dimensional theory then is used to motivate the general divergence theorem. The difficulties in successfully carrying the Henstock-Kurzweil techniques into higher dimensions are described. This leads naturally to a discussion about a number of topics from geometric measure theory: Hausdorff measure, sets of bounded variation, rectifiability, and theorems of De Giorgi and of Federer. From there we are led to the integration theory of W. F. Pfeffer from his treatise [*Derivation and integration*, Cambridge Univ. Press, Cambridge, 2001; MR1816996 (2001m:26018)]. (See also the recent survey [W. F. Pfeffer, Sci. Math. Jpn. **55** (2002), no. 2, 399–425; MR1887074 (2002k:26023)] for a compelling account.) Included too is an integration by parts formula from [T. De Pauw and W. F. Pfeffer, Adv. Math. **183** (2004), no. 1, 155–182; MR2038549 (2004k:35035)].

Finally some applications of the divergence theorem are given. Federer's proof that the Bombieri-De Giorgi-Giusti cone in $\mathbb{R}^8$ is minimal is sketched. A removable singularities theorem for the minimal surface equation is stated.

{For the entire collection see MR2145143 (2006c:00014)}

Brian S. Thomson (Vancouver, BC)

MR2009755 (2004g:26005) 26A24 26A21

Shi, Hongjian (1-LSVL-VIP)

A type of path derivative. (English. English summary)

Real Anal. Exchange **28** (2002/03), *no.* 2, 279–286.

The derivative that the author introduces is a variant of the sequential derivative. Let $\{h_n\}$ be a fixed sequence of real numbers tending to zero. Then, at any point x, the sequential derivative of a function F is defined, if it exists, as the function g given by the limit $g(x) = \lim_{n\to\infty}[F(x+h_n)-F(x)]/h_n$. By using the same sequence at every point one gets an unusual derivative with interesting and curious properties [see M. Laczkovich and G. Petruska, Acta Math. Acad. Sci. Hungar. **38** (1981), no. 1-4, 205–214; MR0634581 (83b:26005)].

In the variant here sequences monitor the derivative in a different way. Let $\{h_n\}$ and $\{k_n\}$ be two fixed sequences of real numbers decreasing to zero. At each point x there should exist two sequences $\{t_n\}$ and $\{s_n\}$ (both may depend on x) with $t_n \in [h_{n+1}, h_n]$ and $s_n \in [k_{n+1}, k_n]$ for sufficiently large n, such that the function g given by the limit $g(x) = \lim_{n\to\infty}[F(x+t_n)-F(x)]/t_n = \lim_{n\to\infty}[F(x)-F(x-s_n)]/s_n$. The author studies a number of implications that such a weak derivative might have including monotonicity properties, Darboux properties, typical properties and measurability properties.

Brian S. Thomson (Vancouver, BC)

MR1939729 (2003h:26006) 26A15 26A21

Marciniak, Mariola (PL-LODZ-CC)**;**
Pawlak, Ryszard J. (PL-LODZM)

On the restrictions of functions, finitely continuous functions and path continuity. (English. English summary)

Tatra Mt. Math. Publ. **24** (2002),, *part I*, 65–77.

A real-valued function defined on the real line is said to be finitely continuous if there is a finite partition of the line into a collection of sets relative to each of which the function is continuous. Continuing the study in [M. Marciniak, Real Anal. Exchange **26** (2000/01), no. 1, 417–420; MR1825520 (2001m:26006)] the authors here obtain a strong result for the continuity properties of a finitely continuous Darboux function.

The second half of the paper is devoted to a somewhat related study of conditions under which a function will be Darboux or almost continuous in the sense of Stallings. These conditions are expressed in the language of path continuity.

Brian S. Thomson (Vancouver, BC)

MR1923649 (2003h:26013) 26A39 26A24 26A45

Bongiorno, B. (I-PLRM); **Di Piazza, L.** (I-PLRM);
Skvortsov, V. A. (RS-MOSC)

On dyadic integrals and some other integrals associated with local systems. (English. English summary)

J. Math. Anal. Appl. **271** (2002), *no.* 2, 506–524.

This article is mostly concerned with the study of Henstock-type integrals associated with dyadic derivatives on the real line and will be of interest to experts in non-absolutely convergent integrals as well as to fans of Haar series. It opens with an interesting general discussion of descriptive integral characterizations and an abstract framework (within the context of local systems and path-systems) for the study of such integrals.

For the dyadic integral studied here, most of the results are comfortably similar to the familiar Henstock treatment of the Denjoy-Perron integrals using variational measures, although the details of the proofs require many adjustments for the dyadic treatment.

Brian S. Thomson (Vancouver, BC)

[References]

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

1. E.S. Bajgogin, On a dyadic Perron integral, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 48 (1993) 25–28. MR1274657 (95b:26012)
2. B. Bongiorno, L. Di Piazza, D. Preiss, Infinite variation and derivatives in $\mathbf{R}^n$, J. Math. Anal. Appl. 224 (1998) 22–33. MR1632942 (99j:28003)
3. B. Bongiorno, L. Di Piazza, V. Skvortsov, The essential variation of a function and some convergence theorems, Anal. Math. 22 (1996) 3–12. MR1384345 (97a:26011)
4. B. Bongiorno, L. Di Piazza, V. Skvortsov, A new full descriptive characterization of the Denjoy-Perron integral, Real Anal. Exchange 21 (1995/96) 656–663. MR1407278 (97h:26008)
5. B. Bongiorno, L. Di Piazza, V. Skvortsov, On variational measures related to some bases, J. Math. Anal. Appl. 250 (2000) 533–547. MR1786079 (2001i:26007)
6. B. Bongiorno, W. Pfeffer, B.S. Thomson, A full descriptive definition of the gage integral, Canad. Math. Bull. 39 (1996) 390–401. MR1426684 (97j:26008)
7. A.M. Bruckner, R.J. O'Malley, B.S. Thomson, Path derivatives: a unified view of certain generalized primitives, Trans. Amer. Math.

Soc. 283 (1984) 97–125. MR0735410 (86d:26007)

8. Z. Buczolich, W. Pfeffer, On absolute continuity, J. Math. Anal. Appl. 222 (1998) 64–78. MR1623859 (99i:28011)
9. W. Cai-shi, D. Chuan-Song, An integral involving Thomson's local systems, Real Anal. Exchange 19 (1993/94) 248–253. MR1268851 (95a:26008)
10. L. Di Piazza, Variational measures in the theory of the integration in R^m, Czechoslovak Math. J. 51 (2001) 95–110. MR1814635 (2001m:28003)
11. V. Ene, Real Functions—Current Topics, Lecture Notes in Math., Vol. 1603, Springer-Verlag, 1995. MR1369575 (96k:26001)
12. V. Ene, Thomson's variational measures, Real Anal. Exchange 24 (1998/99) 523–566. MR1704732 (2000g:26005)
13. Sh. Fu, Path integral: an inversion of path derivatives, Real Anal. Exchange 20 (1994/95) 340–346. MR1313697 (95m:26008)
14. R.A. Gordon, The inversion of approximate and dyadic derivatives using an extension of the Henstock integral, Real Anal. Exchange 16 (1990/91) 154–168. MR1087481 (92b:26015)
15. R.A. Gordon, The Integrals of Lebesgue, Denjoy, Perron, and Henstock, Graduate Studies in Math., Vol. 4, American Mathematical Society, Providence, RI, 1994. MR1288751 (95m:26010)
16. R.A. Gordon, Some comments on an approximately continuous Khintchine integral, Real Anal. Exchange 20 (1994/95) 831–841. MR1348106 (96m:26008)
17. J.M. Jędrzejewski, On limit numbers of real functions, Fund. Math. 83 (1973/74) 269–281. MR0330365 (48 #8702)
18. J.P. Kahane, Une théorie de Denjoy des martingales dyadiques, Enseign. Math. (2) 34 (1988) 255–268. MR0979642 (90h:60042)
19. Y. Kubota, An integral of the Denjoy type, Proc. Japan Acad. 40 (1964) 713–717. MR0178113 (31 #2371)
20. S. Lu, Notes on approximately continuous Henstock integral, Real Anal. Exchange 22 (1996/97) 377–381. MR1433622 (97k:26010)
21. K.M. Ostaszewski, Henstock integration in the plane, Mem. Amer. Math. Soc., Vol. 353, American Mathematical Society, Providence, RI, 1986. MR0856159 (87j:26016)
22. A. Pacquement, Determination d'une fonction au moyen de sa derivée sur un reseau binaire, C. R. Acad. Sci. Paris Sér. A 284 (1977) 365–368. MR0430192 (55 #3199)
23. W.F. Pfeffer, The Riemann Approach to Integration, Cambridge University Press, Cambridge, 1993. MR1268404 (95f:26001)
24. S. Saks, Theory of the Integral, Dover, New York, 1964. MR0167578 (29 #4850)

25. D.N. Sarkhel, On the approximately continuous integrals of Burkill and Kubota, Real Anal. Exchange 23 (1997/98) 735–742. MR1639949 (99g:26021)
26. V.A. Skvortsov, Calculation of the coefficients of an everywhere convergent Haar series, Mat. Sb. 75 (1968) 349–360; Engl. transl. Math. USSR-Sb. 4 (1968) 317–327. MR0226294 (37 #1884)
27. V.A. Skvortsov, Some properties of dyadic primitives, in: New Integrals, Lectures Notes in Math., Vol. 1419, Springer-Verlag, 1988, pp. 167–179. MR1051928
28. V.A. Skvortsov, A variational measure and sufficient conditions for the differentiability of an additive interval function, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 2 (1997) 55–57; Engl. transl. Moscow Univ. Math. Bull. 52 (1997) 38–39. MR1615443
29. V.A. Skvortsov, Variation and variational measures in integration theory and some applications, J. Math. Sci. (NY) 91 (1998) 3293–3322. MR1657287 (99k:26009)
30. V.A. Skvortsov, P. Sworowski, On variational measures associated with approximate differentiation bases, Vestnik Moskov. Univ. Ser. I Mat. Mekh., in press.
31. B.S. Thomson, Real Functions, Lecture Notes in Math., Vol. 1170, Springer-Verlag, 1980. MR0818744 (87f:26001)
32. B.S. Thomson, Derivation bases on the real line, Real Anal. Exchange 8 (1982/83) 67–207, 278–442. MR0700194 (84i:26008b)
33. B.S. Thomson, Some property of variational measures, Real Anal. Exchange 24 (1998/99) 845–854. MR1704758 (2000g:26006)

MR1887855 (2003d:26004) 26A24

Mitra, S.; Mukhopadhyay, S. N. (6-BURD)

Derivates, approximate derivates and porosity derivates of n-convex functions. (English. English summary)

Real Anal. Exchange **27** (2001/02), *no.* 1, 249–259.

A. Khinchin [Fund. Math. **9** (1927), 212–279; JFM 53.0229.01] proved that if a monotonic function has an approximate derivative at a point then it is in fact differentiable there. The argument is not deep although, at first sight, it appears to be so, depending as it does on the notion of Lebesgue density. A closer look reveals that the only property needed is that a set having density 1 at a point cannot have relatively large gaps near the point (i.e., it must be nonporous at the point); then a simple geometric argument gives estimates on the Dini derivates. The discussion by A. M. Bruckner, R. J. O'Malley and B. S. Thomson [Trans. Amer. Math. Soc. **283** (1984), no. 1, 97–125; MR0735410 (86d:26007)] in the setting of path derivatives clarifies this. Nonetheless, Khinchin's observation plays a key role in the study of approximate derivatives; for example, with it and some other properties of approximate derivatives one can prove that an everywhere approximately differentiable function must be differentiable in the ordinary sense on a dense open set.

In the paper under review the authors extend Khinchin's theorem to higher order derivates. They show that if a function is n-convex then the four nth order Peano derivates agree with their corresponding approximate versions. The same is true for several other generalized higher order derivates. Once again it is the notion of porosity that allows the estimates to be made in the main proof.

Brian S. Thomson (Vancouver, BC)

MR1836387 (2002c:26006) 26A21 26A24

Pawlak, Ryszard Jerzy (PL-LODZM)

On additive lattices of some subfamily (B_1^{*+}) of the family DB_1. (English. English summary)

Atti Sem. Mat. Fis. Univ. Modena **49** (2001), *no.* 1, 111–120.

The author introduces and studies a class of functions contained in the class of Darboux Baire 1 functions defined on the real line. The importance of the latter class arises from the fact that many of the functions arising in real analysis, most notably all derivatives and all approximate derivatives, enjoy these properties. Perhaps the earliest of the studies of special subclasses of the Darboux Baire 1 functions is that of R. J. O'Malley [Proc. Amer. Math. Soc. **60** (1976), 187–192; MR0417352 (54 #5405)], which is often cited and used.

The present paper is related to an earlier one by the same author [Tatra Mt. Math. Publ. **19** (2000), part I, 135–144; MR1771030 (2001h:26004)] in which he introduced a slightly larger class of functions.

Brian S. Thomson (Vancouver, BC)

MR1825496 (2002b:26008) 26A45 26A39 26A46

Ene, Vasile

Thomson's variational measure and nonabsolutely convergent integrals. (English. English summary)

Real Anal. Exchange **26** (2000/01), *no.* 1, 35–49.

This article continues some themes developed in an earlier paper by the author [Real Anal. Exchange **25** (1999/00), no. 2, 521–545; MR1778509 (2001i:26009)], along with applications to integrals of the Stieltjes type.

Brian S. Thomson (Vancouver, BC)

MR1778510 (2001i:26002) 26A15

Szyszkowski, Marcin (1-WV)

Symmetrically continuous functions on various subsets of the real line. (English. English summary)

Real Anal. Exchange **25** (1999/00), *no.* 2, 547–564.

A real function f is said to be symmetrically continuous at a point x provided that $\lim_{h\to 0} f(x+h) - f(x-h) = 0$. The normal assumption is that f is defined in a neighborhood of x, although not necessarily at x. There are many questions that have been asked (and answered) about such functions. A survey along with proofs can be found in the reviewer's monograph [*Symmetric properties of real functions*, Dekker, New York, 1994; MR1289417 (95m:26002)].

In this interesting and well-written paper the author considers the notion of symmetric continuity applied to functions whose domain is an arbitrary subset of the real line. This requires some careful attention to detail in the definitions as well as in the proofs. Using clever refinements of known methods, he presents a full account of the questions that arise and extends many of the known results.

Brian S. Thomson (Vancouver, BC)

MR1778509 (2001i:26009) 26A45 26A24 26A46

Ene, Vasile

Thomson's variational measure and some classical theorems. (English. English summary)

Real Anal. Exchange **25** (1999/00), *no.* 2, 521–545.

This article shows how the variational measure of a function can be exploited to prove (and improve) various classical theorems in the theory of real functions of a single real variable. In particular the Denjoy-Young-Saks and de la Vallée Poussin theorems are proved and extended. This is one of the last papers of Vasile Ene—it was completed after his death with the help of the author's wife and very generous and detailed assistance from the referee.

Brian S. Thomson (Vancouver, BC)

MR1760817 (2001f:26010) 26A42 26A39

Pfeffer, Washek F. (1-CAD)

The Lebesgue and Denjoy-Perron integrals from a descriptive point of view. (English. English summary)

Ricerche Mat. **48** (1999), *no.* 2, 211–223 (2000).

The relationship between the Denjoy-Perron and Lebesgue integrals on the real line can be expressed by the assertion that the expression

$$F(x) = \int_a^x f(t)\,dt \ (a \le x \le b)$$

holds if $F'(x) = f(x)$ for a.e. $x \in [a,b]$ and F is AC on $[a,b]$ (for the Lebesgue integral) or if F is ACG_* on $[a,b]$ (for the Denjoy-Perron integral). This ACG_* concept, exploited so well in [S. Saks, *Theory of the integral*, Second revised edition. English translation by L. C. Young. With two additional notes by Stefan Banach, Dover, New York, 1964; MR0167578 (29 #4850)], is not the most successful or transparent way to express the ideas underlying these characterizations. As the author of this article points out, it is "cumbersome and relies completely on the order structure of the reals".

The paper here is a well-written and nicely accessible exposition with new proofs and many clarifications of some recent revisions of these fundamental ideas. Without giving all the details, we can sketch the main themes by indicating that to any function $F\colon [a,b] \to \mathbf{R}$ there is associated a Borel regular measure V_*F on $[a,b]$ called the critical variation of F. If that measure is absolutely continuous with respect to Lebesgue measure, then F is a.e. differentiable and $V_*F(E) = \int_E |F'(x)|\,dx$ for all measurable $E \subset [a,b]$. The elegant proof of this fact here uses Vitali coverings, Baire category, and an appeal to a differentiability theorem of Ward (or Stepanov). Using that, it is relatively easy to establish that the relationship between the Denjoy-Perron and Lebesgue integrals can be expressed by the requirement that the critical variation measure V_*F is absolutely continuous with respect to Lebesgue measure (for the Denjoy-Perron integral) or both finite and absolutely continuous with respect to Lebesgue measure (for the Lebesgue integral). The paper concludes with a general discussion for Borel measures in a metric space of the distinction among the concepts semi-moderated, semi-locally finite, and σ-finite. These concepts intrude here since, if V_*F is absolutely continuous with respect to Lebesgue measure, then it is necessarily semi-moderated. *Brian S. Thomson* (Vancouver, BC)

MR1704732 (2000g:26005) 26A45 26A24 26A46

Ene, Vasile (R-OV)

Thomson's variational measure. (English. English summary)

Real Anal. Exchange **24** (1998/99), *no.* 2, 523–565.

This paper continues the research program that the author, who died on November 11, 1998, has described in his monograph [*Real functions—current topics*, Lecture Notes in Math., 1603, Springer, Berlin, 1995; MR1369575 (96k:26001)]. This paper establishes many delicate relations holding for a function among its differentiation properties, its variational properties, and properties of its associated variational measures. Since all of these require considerable technical details to state, we will only relate a sample that is representative in part. For a real-valued function f defined on an interval and a measurable subset P of that interval the following are equivalent: (i) f is ACG$_*$ on P; (ii) the variational measure μ_f associated with f is absolutely continuous on P; (iii) f is continuous at each point of P and has a finite derivative μ_f-a.e. in P; (iv) f is continuous at each point of P and $\int_E |f'(x)|dx = \mu_f(E)$ for all measurable $E \subset P$. There are many results in the same spirit as this, some expressed, as here, for the ordinary derivative and ordinary variation, and others expressed within the language of local systems (as presented in the reviewer's monograph [*Real functions*, Lecture Notes in Math., 1170, Springer, Berlin, 1985; MR0818744 (87f:26001)]) and some in the setting of approximate derivatives. *Brian S. Thomson* (Vancouver, BC)

MR1396976 (97j:26004) 26A24 26A21

Fejzić, Hajrudin (1-CASSB)**; Rinne, Dan** (1-CASSB)

Peano path derivatives. (English. English summary)

Proc. Amer. Math. Soc. **125** (1997), *no.* 9, 2651–2656.

The notion of a path derivative was introduced by A. M. Bruckner, R. J. O'Malley and B. S. Thomson [Trans. Amer. Math. Soc. **283** (1984), no. 1, 97–125; MR0735410 (86d:26007)]. Suppose that at each point $x \in \mathbf{R}$ there is assigned an appropriate set $E_x \subset \mathbf{R}$ so that a derivative of the form

$$f'_E(x) = \lim_{h\to 0, x+h\in E_x} h^{-1}(f(x+h) - f(x))$$

can be defined for a real function f. Then f'_E is called the path derivative of f relative to the system $E = \{E_x\colon\ x \in \mathbf{R}\}$. Many of the classical generalized derivatives can be expressed in this way. It has been shown that a variety of properties of such derivatives can be obtained directly from a study of the way that the paths intersect, that is from the sets $E_x \cap E_y$ for x close to y.

In the present article the authors apply these ideas to the study of generalized versions of the nth order Peano derivative. The Peano derivative is defined much in the usual way but taken relative to a system of paths. They introduce a simple and elegant intersection condition which is sufficient to assert that if a function is n-times Peano path differentiable then its nth order Peano path derivative is Baire 1 and all lower order derivatives are Baire* 1. This result is applied to approximate Peano derivatives, left and right approximate Peano derivatives, and to their analogues in the sense of $\mathcal{I}$-density. (The $\mathcal{I}$-density is a category analogue of ordinary density due to W. Wilczyński [see, e.g., W. Poreda, E. Wagner-Bojakowska and W. Wilczyński, Fund. Math. **125** (1985), no. 2, 167–173; MR0813753 (87b:54034)].) *Brian S. Thomson* (Vancouver, BC)

[References]

Note: This list, extracted from the PDF form of the original paper, may contain data conversion errors, almost all limited to the mathematical expressions.

1. A. M. Bruckner, R. J. O'Malley and B. S. Thomson, *Path Derivatives: A Unified View of Certain Generalized Derivatives,* Trans. Amer Math. Soc. Vol. 283 **1** (1984), 97-125. MR0735410 (86d:26007)
2. H. Fejzić, *The Peano Derivatives,* Ph.D. Dissertation, Michigan State University (1992).
3. E. Lazarow, *On the Baire Class of* I *-approximate Derivatives,* Proc. Amer. Math. Soc. Vol. 100 **4** (1987), 669-674. MR0894436 (88h:26001)
4. W. Poreda, E. Wagner-Bojakowska, and W. Wilczyński, *A Category Analogue of the Density Topology*, Fund. Math. Vol. 128 (1985), 167-173. MR0813753 (87b:54034)

MR1634710 (99d:26014) 26A45 26A24

Sarkhel, D. N. (6-KALY)

Variational characterisation of functions having Lebesgue integrable derivatives. (English. English summary)

Indian J. Math. **38** (1996), *no.* 3, 223–235.

Summary: "We show that the finiteness of a certain strong [weak] core variation of a measurable function on a linear set is both necessary and sufficient for the function to have a Lebesgue integrable ordinary [approximate] derivative a.e., and that the integrals of the derivative and its absolute value are representable by these variations."

Brian S. Thomson (Vancouver, BC)

MR1460985 (98g:26007) 26A24

Bullen, P. S. (3-BC); **Sarkhel, D. N.** (6-KALY)

Properties of derivative-like functions. (English. English summary)

Real Anal. Exchange **22** (1996/97), *no.* 2, 740–759.

This article studies properties of a class of generalized derivatives. The derivatives are all of the type that can be defined using a local system. For background on local systems one should consult the monograph of V. Ene [*Real functions—current topics*, Lecture Notes in Math., 1603, Springer, Berlin, 1995; MR1369575 (96k:26001)] or that of the reviewer [*Real functions*, Lecture Notes in Math., 1170, Springer, Berlin, 1985; MR0818744 (87f:26001)]. In addition to the usual class of derivatives and extreme derivates that can be defined in terms of a local system, the authors also define what they call a "derivative-like" function which is related. Many properties of these derivatives are developed.

As is usual in studies of this nature, the conditions that are imposed on the local system are intersection conditions and porosity conditions. The authors have introduced as well interesting new conditions which they call "the interlocking property" and NPG (non-porous in the generalized sense). With the assumption of these two properties they are able to show that all derivative-like functions have versions of properties (known for the ordinary and approximate derivatives) due to Z. Zahorski [Trans. Amer. Math. Soc. **69** (1950), 1–54; MR0037338 (12,247c)], C. E. Weil [Proc. Amer. Math. Soc. **20** (1969), 487–490; MR0233944 (38 #2265)], and R. J. O'Malley [Proc. Amer. Math. Soc. **54** (1976), 122–124; MR0390143 (52 #10969)].

Brian S. Thomson (Vancouver, BC)

MR1460977 (98g:26008) 26A24

Ene, Vasile (R-OV)

Local systems and Taylor's theorem. (English. English summary)

Real Anal. Exchange **22** (1996/97), *no.* 2, 638–644.

Taylor's theorem, with one of the standard forms of the remainder, is the formula

$$F(b)=\sum_{i=0}^{n}\frac{D^iF(a)}{i!}(b-a)^i+\frac{D^{(n+1)}F(\xi)}{(n+1)!}(b-a)^{n+1},$$

where ξ is some point between a and b. It is of some interest to know if this formula can be interpreted in any weaker senses using various generalized derivatives. The author gives conditions under which this is possible for a class of derivatives taken in the setting of a local system. The conditions are expressed as intersection conditions, conditions that will be familiar to readers who have studied such systems. For the necessary background on local systems one should consult the monograph of the author [*Real functions—current topics*, Lecture Notes in Math., 1603, Springer, Berlin, 1995; MR1369575 (96k:26001)] or that of the reviewer [*Real functions*, Lecture Notes in Math., 1170, Springer, Berlin, 1985; MR0818744 (87f:26001)].

Brian S. Thomson (Vancouver, BC)

MR1403216 (97m:26010) 26A39 26A45 42A16

Mukhopadhyay, S. N. (6-BURD)**;**
Mukhopadhyay, S. K. (6-BURD)

A generalized integral with applications to trigonometric series. (English. Russian summary)

Anal. Math. **22** (1996), *no.* 2, 125–146.

A real function f is ACG on an interval if that interval can be covered by a sequence of sets, on each of which f is absolutely continuous in a certain sense. This definition was given by S. Saks [*Theory of the integral*, second revised edition, Stechert, New York, 1937; Zbl 017.30004], along with applications to the study of approximate derivatives and to a characterization of the general integral of Denjoy. Here the authors generalize this notion to a higher order version. A function f is $\mathrm{AC}_2\mathrm{G}$ on an interval if that interval can be covered by a sequence of sets, on each of which f is AC_2, where that notion is defined using second order divided differences.

This concept is used to give a descriptive definition of a second order version of the Denjoy integral. Some properties of that integral are developed, including an integration by parts formula, and comparisons are made with other similar second order integrals such as the Cesàro-Perron and symmetric Cesàro-Perron integrals of J. C. Burkill.

The main motivation for introducing this integral has been for applications to a.e. convergent trigonometric series. It is shown that such a series, under an extra (strong) hypothesis, can be expressed as a Fourier series relative to this generalized integral. The hypothesis here is that the Riemann function for the series (i.e., the function obtained by two formal integrations) is AC_2G. It seems this condition would be difficult to verify directly from properties of the series. One set of conditions is presented that does allow this verification. A number of unsolved problems remain. The most critical, it would seem, is the question as to whether an everywhere convergent trigonometric series has a Riemann function that is AC_2G.

Brian S. Thomson (Vancouver, BC)

MR1377551 (97b:26006) 26A39 26B15

Lin, Ying-Jian

On the generalized convergence theorems for Thomson's $\mathcal{B}$-integral on $\mathbf{R}^m$.

Real Anal. Exchange **21** (1995/96), *no.* 1, 365–379.

The author defines and investigates an integral of Henstock type defined relative to an abstract differentiation basis $\mathcal{B}$ in $\mathbf{R}^m$. The basis is assumed to have five properties that are just enough to develop an adequate integration theory. Let $f_n \to f$ a.e. on an interval I_0 in $\mathbf{R}^m$ so that $\{f_n\}$ are pointwise bounded there. The main intent of the paper is to investigate conditions under which one can assert that $\mathcal{B}\int_{I_0} f = \lim_{n\to\infty} \mathcal{B}\int_{I_0} f_n$. The author gives five equivalent conditions on the sequence $\{f_n\}$ so that this is true. One is an equi-integrability condition on $\{f_n\}$ and several concern ways in which the corresponding primitive functions $\{F_n\}$ are uniformly absolutely continuous in certain senses.

For the most part, both in techniques and ideas, this is an extension to a more general setting of material that can be found for the one-dimensional integral in papers by S. P. Lu and P. Y. Lee [Real Anal. Exchange **16** (1990/91), no. 2, 537–545; MR1112049 (92f:26014)] and R. A. Gordon [Real Anal. Exchange **18** (1992/93), no. 1, 261–266; MR1205521 (94a:26016)] and also, in a different m-dimensional version, in work of J. Kurzweil and J. Jarník [Real Anal. Exchange **17** (1991/92), no. 1, 110–139; MR1147361 (93d:26006)].

Brian S. Thomson (Vancouver, BC)

MR1377536 (97b:26007) 26A39 26A45

Skvortsov, Valentin A. (RS-MOSC)

Continuity of δ-variation and construction of continuous major and minor functions for the Perron integral. (English. English summary)

Real Anal. Exchange **21** (1995/96), *no.* 1, 270–277.

The Henstock integral is well known to be equivalent to Perron's integral. The connection is almost immediate. In one direction one uses the existence of the Perron integral to obtain a major/minor function pair that can be used to obtain an estimate on the Riemann sums. In the converse direction one uses the existence of the Henstock integral to obtain major and minor functions from the variation. The only difficulty here is that this method does not give continuous major and minor functions. Hence all that is proved is that the Henstock integral is equivalent to a version of the Perron integral that does not specify continuity for the major and minor functions. To obtain equivalence with Perron's original definition (that does employ continuous majorants and minorants) requires an appeal to deeper methods.

In this article the author shows how to construct continuous major and minor functions directly from the variation. He first shows that the variation taken relative to a fixed gauge δ is continuous. While this variation need not be additive, a clever device is used to obtain one from it that is additive and continuous.

These techniques should be of considerable interest to individuals working on generalized versions of these integrals. In particular this offers the most direct proof that the Henstock integral is equivalent to the classical Perron integral. *Brian S. Thomson* (Vancouver, BC)

MR1257105 (95k:26005) 26A24 26A39

Fu, Shushang (PRC-FZHU)

λ-power integrals on the Cantor type sets. (English. English summary)

Proc. Amer. Math. Soc. **123** (1995), *no.* 9, 2731–2737.

Let C be a Cantor set of Hausdorff dimension λ $(0<\lambda<1)$ and whose λ-dimensional Hausdorff measure is finite and positive, let D denote the countable set of endpoints of the intervals complementary to C and let F denote a continuous, singular function supported on C, so that $F'(x)=0$ at each $x\notin C$. The problem posed by the author is to determine F from its derivative in a generalized sense. For each

$x \in C \setminus D$ a λ-power derivative is defined as

$$F'_\lambda(x) = \lim_{m\to\infty} \frac{F(b_m^{j(m)}) - F(a_m^{j(m)})}{(b_m^{j(m)} - a_m^{j(m)})^\lambda}$$

where the limit is taken following a net structure of intervals shrinking to x with endpoints in D. The term "dyadic" is used throughout, but the net is not the one usually called by that name.

The main theorem asserts that if $F'_\lambda(x) \geq 0$ at each $x \in C \setminus D$ except possibly for some countable set then F is nondecreasing. This is used, in the obvious way, to define a descriptive integral that inverts exact λ-power derivatives of singular functions. The author then studies the special case of a function f that vanishes off C and is continuous relative to C. For this the construction of a primitive function F with $F'_\lambda(x) = f(x)$ for $x \in C \setminus D$ and $F'(x) = 0$ for $x \notin C$ is readily given. *Brian S. Thomson* (Vancouver, BC)

MR1305251 (95m:26018) 26A48 26A24

Filipczak, Tomasz (PL-LODZ-IM)

Monotonicity theorems for the I-proximal local system.

Demonstratio Math. **27** (1994), *no.* 2, 517–520.

In an earlier paper [Real Anal. Exchange **19** (1993/94), no. 1, 114–120; MR1268836 (95b:26015)] the author established some delicate general conditions under which the monotonicity of a real function could be deduced from assertions about certain generalized derivates. Here he shows that these conditions are met for a derivate defined using a notion of sparseness arising in the study of $\mathcal{I}$-density. This latter concept was first introduced by W. Wilczyński in 1985 as a category analogue of density. The best reference for a background to questions in $\mathcal{I}$-density is the monograph by K. Ciesielski, L. M. Larson and K. M. Ostaszewski [Mem. Amer. Math. Soc. **107** (1994), no. 515, xiv+133 pp.; MR1188595 (94f:54035)].

Brian S. Thomson (Vancouver, BC)

MR1268851 (95a:26008) 26A39

Wang, Cai Shi (PRC-NWTE); **Ding, Chuan Song** (PRC-NWTE)

An integral involving Thomson's local systems.

Real Anal. Exchange **19** (1993/94), *no.* 1, 248–253.

An integral of generalized Riemann type is defined relative to a differentiation basis on the real line. This is defined in the setting of "local systems" that are filtering, bilateral and satisfy an intersection condition. A descriptive characterization of Luzin type is given in terms of derivatives taken with respect to the local system. The integral includes as special cases the original Henstock-Kurzweil integral, an approximately continuous integral of Burkill and an integral based on the dyadic derivative. *Brian S. Thomson* (Vancouver, BC)

MR1268837 (95f:26011) 26A39 26A42 28A25

Henstock, R. (4-ULST)

Measure spaces and division spaces. (English. English summary)

Real Anal. Exchange **19** (1993/94), *no.* 1, 121–128.

The generalized Riemann integral, long associated with the author of this article, has an advantage over the Lebesgue theory in that it can be constructed in a setting with not too much structure. The Lebesgue integral requires, first, the construction of a measure space. If a measure space is already given then, under certain hypotheses, the Lebesgue integral can be obtained as a generalized Riemann integral. The article gives two ways this can be done and proves the equivalences. The proofs are self-contained but much of the discussion will be easier to follow for readers of the author's treatise [*The general theory of integration*, Oxford Univ. Press, New York, 1991; MR1134656 (92k:26011)]. *Brian S. Thomson* (Vancouver, BC)

MR1268836 (95b:26015) 26A48

Filipczak, Tomasz (PL-LODZ)

Monotonicity theorems for some local systems.

Real Anal. Exchange **19** (1993/94), *no.* 1, 114–120.

D. N. Sarkhel and A. K. De [J. Austral. Math. Soc. Ser. A **31** (1981), no. 1, 26–45; MR0622811 (82h:26014)] proved the following lemma: Let $a \in A \subset [a,b]$, $B = [a,b] \setminus A$ and suppose that ${}_{+}\underline{d}^{i}(A,x) > 0$ for $x \in A$, $x \neq b$, and that ${}_{-}\underline{d}^{i}(B,x) > 0$ for $x \in B$. Then $B = \varnothing$. (Here ${}_{+}\underline{d}^{i}$ and ${}_{-}\underline{d}^{i}$ denote the right and left lower inner densities.) In the article under review this lemma is formulated in the abstract setting of local systems. A local system is said to satisfy condition (SD) if the analogue of the Sarkhel-De lemma holds for the system. Several monotonicity theorems are established for systems having this property. Finally it is shown that if a local system is bilateral and satisfies an intersection condition then it satisfies (SD) (but not necessarily conversely). Thus, many familiar systems have this property.

Brian S. Thomson (Vancouver, BC)

MR1256057 (94m:26015) 26A39

Li, Bao Ling (PRC-NWTE)

Integration by parts for the SC_n P**-integral.**

Proceedings of the International Conference on Functional Analysis and Global Analysis (Quezon City, 1992).

Southeast Asian Bull. Math. **1993,** *Special Issue*, 73–77.

The integral in the title of the article refers to a scale of integrals introduced by P. S. Bullen and C. M. Lee [Canad. J. Math. **25** (1973), 1274–1284; MR0409737 (53 #13489)] generalizing the original symmetric Cesàro-Perron (SCP) integral of J. C. Burkill [Proc. London Math. Soc. (3) **1** (1951), 46–57; MR0042533 (13,126e)]. The main purpose is to present an integration by parts formula for these integrals. The strategy is to give an equivalent variational formulation of the integrals, following ideas of R. Henstock [see, e.g., Proc. London Math. Soc. (3) **10** (1960), 281–303; MR0121460 (22 #12198)], and then use the techniques which work for ordinary (nonsymmetric) integrals. One should recall that the proof for integration by parts in the original paper of Burkill was incorrect. A correct proof, given by V. A. Sklyarenko [Mat. Sb. (N.S.) **112(154)** (1980), no. 4(8), 630–646; MR0587041 (81k:26009)], was long and difficult. As this article contains no proofs or indications of proofs, it remains to be seen whether this program will be successful.

{For the entire collection see MR1256048 (94g:00019)}

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