Recovering a Function from a Dini Derivative

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1. INTRODUCTION. One of the central questions in Lebesgue's masterpiece [**6**] is that of determining a function, up to an additive constant, if one of the four Dini derivatives of the function is known. This problem is, of course, best known in its less general version: determine F up to a constant given the value of $F'(x)$ at every point x. As all students of the calculus will remember, the formula

$$F(b) - F(a) = \int_a^b F'(x)\,dx$$

provides a clear answer if we can assume that F' is Riemann integrable. Students of analysis will learn that if F' is Lebesgue integrable the same formula can be used, interpreting the integral in this more general sense. A full resolution of the problem requires a more general integral still, that of Denjoy and Perron (known frequently now as the Henstock-Kurzweil integral).

The main question of this paper is, as it was for Lebesgue, whether a function can be recovered as an indefinite integral of one of its Dini derivatives—that is, when does the formula

$$F(b) - F(a) = \int_a^b D^+F(x)\,dx$$

hold? We shall need to go beyond the Riemann integral, yet we remain committed to answering this problem by using some kind of a Riemann sum process. The investigation is in certain respects similar to the case involving the derivative F', but the path is more interesting, varied, and even surprising at points.

Let us recall some details about Dini derivatives. We can focus just on the right-hand derivatives and use the familiar notation

$$D^+F(x) = \limsup_{h\to 0+} \frac{F(x+h) - F(x)}{h}, \qquad D_+F(x) = \liminf_{h\to 0+} \frac{F(x+h) - F(x)}{h}$$

as long as the domain of F contains a nondegenerate interval of the form $[x, y]$. These simple facts will be used:

$$D_+F(x) \le D^+F(x), \tag{1}$$

$$D^+F(x) = -D_+[-F(x)], \tag{2}$$

$$D^+[F_1 + F_2](x) \le D^+F_1(x) + D^+F_2(x). \tag{3}$$

Monotonicity Theorem. *If F is continuous on an interval I and $D^+F(x) \ge 0$ at each interior point x of I, then F is nondecreasing on I.*

We now discuss the highlights of Lebesgue's investigation of this problem. They are captured by the following two theorems. For simplicity, these and other theorems are stated for a function F whose domain is the entire real line. This is no real restriction since a function F whose domain is a closed, bounded interval $[a, b]$ can be extended to $\mathbb{R}$ by setting $F(x) = F(a)$ for $x < a$ and $F(x) = F(b)$ for $x > b$. In this way, if F is continuous on $[a, b]$, then its extension to $\mathbb{R}$ is also continuous, and if $D^+F(x)$ is finite for each x in $[a, b)$, then $D^+F(x)$ is finite on $\mathbb{R}$.

Theorem 1. *If F is a continuous function with a finite Dini derivative $D^+F(x)$ at every point x of $\mathbb{R}$, then F is determined up to an additive constant by the values of $D^+F(x)$.*

Theorem 2. *If F is a continuous function that has a finite Dini derivative $D^+F(x)$ at every point x of $\mathbb{R}$, then*

$$F(b) - F(a) = \int_a^b D^+F(x)\,dx \tag{4}$$

for each interval $[a, b]$, with the understanding that an appropriate choice of integral must be used.

Theorem 1 may well be a bit surprising: we fully expect a function to be determined by the value of its derivative, but can a single Dini derivative carry that much information? Lebesgue's proof is entirely elementary and uses no ideas not known to Dini himself. Take two continuous functions F and G for which $D^+F(x) = D^+G(x)$ at every point (remembering that these values are finite). Obtain, using (3), that

$$0 = D^+F(x) - D^+G(x) \leq D^+[F - G](x). \tag{5}$$

The monotonicity theorem then shows that the continuous function $F - G$ is nondecreasing. Reverse the roles of F and G to show that $G - F$ is also nondecreasing and conclude that F and G differ by a constant, as desired. (Lebesgue's proof is almost the same but uses a geometric argument similar to Rolle's theorem rather than invoking a monotonicity result.)

As Lebesgue points out, the finiteness of the Dini derivatives plays a role in the argument, since the subtraction step in (5) could not be performed if the values were not all finite. (In fact the theorem would fail if the Dini derivative were infinite at uncountably many points.)

So far this is pretty easy. But it is the second theorem that takes us into a detailed analysis and a long trip to find out what mode of integration will work. Lebesgue shows that if the Riemann integral can be used, then a very simple argument establishes the needed identity (4).

Here is his argument, loosely translated.

Theorem 3. *If F is a continuous function that has a finite Dini derivative $D^+F(x)$ at every point x of $\mathbb{R}$, then*

$$\underline{\int_a^b} D^+F(x)\,dx \leq F(b) - F(a) \leq \overline{\int_a^b} D^+F(x)\,dx \tag{6}$$

for each interval $[a, b]$, where the integrals are the lower and upper Riemann integrals, respectively.

Proof. Divide the interval into subintervals. If $[c, d]$ is one of those subintervals and ℓ and L are the infimum and supremum of $D^+F(x)$ in that interval, then

$$\ell \leq D^+F(x) \leq L$$

and the monotonicity theorem gives

$$\ell(d-c) \leq F(d) - F(c) \leq L(d-c).$$

Sum the inequalities for each of the subintervals, and let the intervals tend to zero to obtain (6). ∎

Now the conclusion of Theorem 2 is immediate, provided we can assume that $D^+F(x)$ is Riemann integrable. The upper and lower integrals are exactly the same, so equal to $F(b) - F(a)$. But Theorem 3 is quite useless for this purpose if the Dini derivative is not Riemann integrable.

Much of the rest of Lebesgue's treatise is devoted to obtaining Theorem 2 under more general integrability conditions. For his own integral (applied to what we would call "Lebesgue integrable functions"—he referred to them as "*fonctions sommables*") a considerable amount of further analysis is needed. After that Lebesgue had no alternative version of Theorem 3 that could work for more general integrals, hence had to develop quite different methods. To handle the problem in full generality required a lengthy excursion into the arcana of Denjoy's totalization methods.

By 1928, when the second edition of Lebesgue's treatise [**6**] was published, there remained many mathematicians who had barely recovered from the shock of learning the new methods of integration introduced in the first edition of 1903. One can only imagine how they would have received the added chapter on totalization.

But there is indeed an exact duplicate of Theorem 3 available that applies to both the Lebesgue integral and the Denjoy-Perron integral. We shall present an elementary proof of a theorem of Saks [**8**, pp. 204–205], albeit expressed in different language, that allows for a simplification of much of Lebesgue's analysis.

The proof of Saks's theorem in [**8**] uses the apparatus of the Perron integral. Since the hypotheses of the theorem are closely aligned with the Perron process, the proof there is not difficult. There seems no longer to be much interest in the Perron integral, and it will no doubt eventually disappear from human memory, a fate that Denjoy would have insisted it richly deserved.

An equally transparent proof of this theorem using the Lebesgue or Denjoy methods is probably not feasible. We show that a proof using the methods of the Henstock-Kurzweil integral can be presented that is entirely elementary and geometrically simple, by basing the argument on a covering lemma first introduced in Hagood [**2**].

We should point out that the methods that Lebesgue used in his analysis for the Lebesgue integrable case [**6**, pp. 176–178] depend on a closely related covering lemma. If a collection $\mathcal{C}$ of intervals has the property that at each point x of an interval $[a, b]$ there is associated an interval $[x, y]$ in $\mathcal{C}$, then from $\mathcal{C}$ can be extracted a transfinite sequence of intervals, ordered from left to right, that are nonoverlapping and cover $[a, b]$. Such a sequence is called a *Lebesgue chain*. It was the existence of such a chain that allowed Lebesgue himself to exploit the geometry of one-sided derivatives and Dini derivatives. Our simpler covering lemma can often replace the use of the Lebesgue chain and is especially useful in the context of Riemann-type integrals.

2. COVERING LEMMAS.

A *covering relation* β is a collection of pairs $(x, [s, t])$ with $s < t$ and x in $[s, t]$. We say that β is a *full cover* of an interval $[a, b]$ if there is a positive function δ defined on $[a, b]$ such that a pair $(x, [s, t])$ belongs to β whenever $[s, t]$ is a subinterval of $[a, b]$ for which $x \leq t < x + \delta(x)$ and $x - \delta(x) < s \leq x$.

This concept is intimately connected to both derivatives and integrals. Suppose that F is differentiable, that $F'(x) = g(x)$ at every point x, and that $\varepsilon > 0$. Then it is easy to check that the covering relation

$$\beta = \left\{(x, [s, t]) : g(x) - \varepsilon < \frac{F(t) - F(s)}{t - s} < g(x) + \varepsilon\right\} \tag{7}$$

is a full cover of any closed, bounded interval. Moreover, the converse is true: if for each $\varepsilon > 0$ the collection defined by (7) is a full cover of every closed, bounded interval, then F is differentiable with $F'(x) = g(x)$ everywhere.

The connection between full covers and integrals arises because every full cover of an interval contains a partition of that interval. This is the basis for the Henstock-Kurzweil theory of integration, which has introduced new and simple methods into the studies of the integrals of Lebesgue, Denjoy, and Perron on the real line.

If

$$\pi = \{(x_i, [t_{i-1}, t_i]) : i = 1, 2, \ldots, n\},$$

where $a = t_0 < t_1 < \cdots < t_n = b$ and x_i lies in $[t_{i-1}, t_i]$ for each i, then π is said to be a *partition* of $[a, b]$. For full covers the covering lemma that is most frequently applied and is most broadly known, due to Pierre Cousin, asserts the existence of partitions.

Lemma 4 (Cousin). *Every full cover of an interval* $[a, b]$ *contains a partition of every subinterval of* $[a, b]$.

As we are concerned here with Dini derivatives, we consider what covering properties are naturally associated with these derivatives. Let F be continuous, let $g(x) = D^+F(x)$ be finite everywhere, and let $\varepsilon > 0$. Then by definition at every point x there is a positive number $\delta(x)$ so that

$$\frac{F(t) - F(x)}{t - x} < g(x) + \varepsilon$$

for every t in $(x, x + \delta(x))$. For each such t we can use the continuity of F to select a positive number $\eta(x, t)$ with the property that

$$\frac{F(t) - F(s)}{t - s} < g(x) + \varepsilon$$

whenever s belongs to $(x - \eta(x, t), x]$. Thus the collection

$$\beta = \left\{(x, [s, t]) : \frac{F(t) - F(s)}{t - s} < g(x) + \varepsilon\right\} \tag{8}$$

is a covering relation that enjoys the third property in the following definition, a property that is an unusual and subtle modification of our full covering property.

Definition 5. A covering relation β is a *right full cover* of an interval $[a, b]$ if for each x in $[a, b]$ there is a positive number $\delta(x)$ with the following properties:

1. every pair $(a, [a, t])$ for which $a < t < a + \delta(a)$ belongs to β;
2. every pair $(b, [s, b])$ for which $b - \delta(b) < s < b$ belongs to β; and

3. for each x in (a, b) and t satisfying $x < t < x + \delta(x)$ there exists a positive number $\eta(x, t)$ such that $(x, [s, t])$ belongs to β whenever $x - \eta(x, t) < s \leq x$.

The right full covers express only one part of the nature of the Dini derivative. Since $D^+F(x)$ is defined by a limit superior, a change in the direction of the inequality in (8) does not give a symmetric covering property. Again let F be continuous, let $g(x) = D^+F(x)$ be finite everywhere, and let $\varepsilon > 0$. Then for every point x there is at least one point $r(x)$ with $r(x) > x$ such that

$$\frac{F(r(x)) - F(x)}{r(x) - x} > g(x) - \varepsilon.$$

We can, as before, use the continuity of F to select $\ell(x)$ with $\ell(x) < x$ such that

$$\frac{F(r(x)) - F(s)}{r(x) - s} > g(x) - \varepsilon$$

for all s in $(\ell(x), x]$. Thus the collection

$$\beta = \left\{(x, [s, t]) : \frac{F(t) - F(s)}{t - s} > g(x) - \varepsilon\right\} \tag{9}$$

is a covering relation that has property three in the next definition, a property that is quite a bit weaker than the right full covering property.

Definition 6. A covering relation β is a *right adequate cover* of an interval $[a, b]$ if for each x in $[a, b]$ there are points $\ell(x)$ and $r(x)$ satisfying $\ell(x) < x < r(x)$ with the following properties:

1. $r(a) \leq b$ and $(a, [a, r(a)])$ belongs to β;
2. $\ell(b) > a$ and every pair $(b, [s, b])$ for which $\ell(b) < s < b$ belongs to β; and
3. for each x in (a, b) it is true that $a < \ell(x) < r(x) < b$ and that $(x, [s, r(x)])$ belongs to β whenever $\ell(x) < s \leq x$.

Observe that every right full cover of $[a, b]$ is a right adequate cover. For these one-sided versions we have the following covering lemma adapted from Hagood [**2**]. Note that the claim is the existence of at least one partition of the entire interval $[a, b]$, not of partitions of all subintervals of $[a, b]$ as in Cousin's lemma.

Lemma 7. *Every right adequate cover of an interval* $[a, b]$ *contains a partition of* $[a, b]$.

Proof. The proof is essentially that in [**2**] modified to suit the current terminology. We suppose that β is a right adequate cover of $[a, b]$ and that $\ell(x)$ and $r(x)$ are the functions that are required by the defining property. Let A be the set of all points x $(a \leq x \leq b)$ for which β contains a partition of $[a, x]$. Certainly $r(a)$ belongs to A. Let $z = \sup A$. If $z < b$, then there must be a point a_1 in $A \cap [\ell(z), z]$. The pair $(z, [a_1, r(z)])$ can be added to a partition of $[a, a_1]$ in β, resulting in a partition of $[a, r(z)]$ and establishing that $r(z)$ lies in A. Since $z < r(z)$, this contradicts the fact that $z = \sup A$. Thus $z = b$, so either b is a member of A, in which event we are done, or there is a point a_1 in $A \cap [\ell(b), b)$. In the latter case, the pair $(b, [a_1, b])$ can be added to a partition of $[a, a_1]$ in β to provide a partition of $[a, b]$. ∎

3. INTEGRALS. We recall the familiar definition of integral using full covers. Given a finite-valued function g on an interval $[a, b]$ and a partition π of $[a, b]$, say

$$\pi = \{(x_i, [t_{i-1}, t_i]) : i = 1, 2, \ldots, n\},$$

we write

$$S(g, \pi) = \sum_{i=1}^{n} g(x_i)(t_i - t_{i-1})$$

for the usual Riemann sum corresponding to the partition.

There are two key features of full covers that permit the development of an integration theory using Riemann sums: every full cover of an interval contains a partition of that interval (i.e., Cousin's lemma), and the intersection of any two full covers is again a full cover.

Definition 8. Let g be a finite-valued function defined on an interval $[a, b]$. Then the *upper integral of g over* $[a, b]$ is defined by

$$\overline{\int_a^b} g(x)\, dx = \inf_{\beta} \sup_{\pi \subset \beta} S(g, \pi),$$

where the supremum is taken over all partitions π of $[a, b]$ contained in a given full cover β of $[a, b]$ and the infimum extends over all such β. The *lower integral of g over* $[a, b]$ is defined by

$$\underline{\int_a^b} g(x)\, dx = \sup_{\beta} \inf_{\pi \subset \beta} S(g, \pi),$$

where the infimum is taken over all partitions π of $[a, b]$ contained in a given full cover β of $[a, b]$ and the supremum extends over all such covers β.

As usual in such theories, g is integrable over $[a, b]$ if the upper and lower integrals agree and are finite, in which case the integral is their common value and is denoted by $\int_a^b g(x)\, dx$. We do not develop the theory here but merely point out that this integral is well known as the Henstock-Kurzweil integral, is equivalent to the Denjoy-Perron integral, and includes both Riemann's and Lebesgue's integrals as special cases. The earliest elementary exposition of the integral was by Henstock [**3**], whose formulation used a familiar ε, δ-approach. Most accounts of this integral are similar to one another and do not use the upper/lower development that we do here. (Leader [**5**] and Lee and Zhao [**7**] give elementary developments of this same integral using the upper and lower integral approach.) Readers familiar with the Darboux theory of integration should have no trouble with these notions.

4. A THEOREM OF SAKS. In this section we exploit Lemma 7 in order to present a conceptually simple proof of our version of a result first proved in the setting of the Perron integral (see Saks [**8**, pp. 204–205]). From that we shall obtain, in Theorem 10, a direct generalization of Theorem 3.

Theorem 9. *Suppose that F is a function that is continuous on the left at every point of $\mathbb{R}$ and that g is a finite-valued function. If $D^+F(x) \geq g(x)$ at every point x, then*

$$F(b) - F(a) \geq \underline{\int_a^b} g(x)\,dx \tag{10}$$

for each interval $[a, b]$. If $D_+F(x) \leq g(x)$ at every point x, then

$$F(b) - F(a) \leq \overline{\int_a^b} g(x)\,dx \tag{11}$$

for each interval $[a, b]$.

Proof. Let $\varepsilon > 0$, and let β be an arbitrary full cover of $[a, b]$. We show that there must exist at least one partition π of $[a, b]$ contained in β for which

$$S(g, \pi) \leq F(b) - F(a) + \varepsilon(1 + b - a).$$

From this inequality (10) follows. The other inequality (11) has a similar proof.

Let β_1 signify the collection of all members $(x, [s, t])$ of β for which $a \leq x < b$ and

$$\frac{F(t) - F(s)}{t - s} > g(x) - \varepsilon, \tag{12}$$

together with those members $(b, [s, b])$ for which

$$F(b) - F(s) > -\varepsilon + (g(b) - \varepsilon)(b - s). \tag{13}$$

We claim that β_1 is a right adequate cover of $[a, b]$. Granting that claim for the moment, we then know that β_1 must include a partition π of the interval $[a, b]$ (Lemma 7). Write π as

$$\pi = \{(x_i, [t_{i-1}, t_i]) : i = 1, 2, \ldots, n\}.$$

Then

$$\begin{aligned} S(g, \pi) &= \sum_{i=1}^{n} g(x_i)(t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} (g(x_i) - \varepsilon)(t_i - t_{i-1}) + \varepsilon(b - a) \\ &\leq \sum_{i=1}^{n} [F(t_i) - F(t_{i-1})] + \varepsilon(1 + b - a) \\ &= F(b) - F(a) + \varepsilon(1 + b - a). \end{aligned}$$

This is precisely the inequality that we require, so the proof is completed by checking that β_1 is a right adequate cover of $[a, b]$.

Since β is a full cover of $[a, b]$, we can select a positive number $\delta(x)$ for each point x of $[a, b]$ in such a way that $(a, [a, t])$ is in β when $a < t < a + \delta(a)$, $(b, [s, b])$ is in β when $b - \delta(b) < s < b$, and $(x, [s, t])$ is in β when $x - \delta(x) < s < t < x + \delta(x)$.

For each x in $[a, b)$ we know that $D^+F(x) \geq g(x)$. This ensures that we can select a point $r(x)$ satisfying $x < r(x) < x + \delta(x)$ and $r(x) \leq b$ with the property that

$$\frac{F(r(x)) - F(x)}{r(x) - x} > g(x) - \varepsilon.$$

If $x > a$, then by the left continuity of F at x there must be a number $\ell(x)$ with $x - \delta(x) < \ell(x) < x$ such that the inequality

$$\frac{F(r(x)) - F(s)}{r(x) - s} > g(x) - \varepsilon \tag{14}$$

also holds whenever $\ell(x) \leq s \leq x$. Inequality (14), in tandem with inequality (12) used to define β_1 and the choices of $\ell(x)$ and $r(x)$, shows that β_1 satisfies the definition of a right adequate cover of $[a, b]$, at least at all points x of $[a, b)$.

There remains only to define $\ell(b)$. Appealing to the left continuity of F at b, we select $\ell(b)$ with $b - \delta(b) < \ell(b) < b$ in such a way that

$$F(b) - F(s) > -\varepsilon/2$$

whenever $\ell(b) \leq s \leq b$ and, at the same time, in such a way that

$$\varepsilon/2 > (g(b) - \varepsilon)(b - s)$$

if $\ell(b) \leq s \leq b$. By so doing we make certain that

$$F(b) - F(s) > -\varepsilon + (g(b) - \varepsilon)(b - s) \tag{15}$$

when $\ell(b) \leq s \leq b$, as we require.

For this choice of $\ell(b)$ inequality (15), together with inequality (13), then completes the proof that β_1 is a right adequate cover of $[a, b]$. ∎

From Saks's theorem we deduce immediately a generalization of Theorem 3, now valid for the upper and lower integrals of section 3.

Theorem 10. *If F is a continuous function that has a finite Dini derivative $D^+F(x)$ at every point x of $\mathbb{R}$, then*

$$\underline{\int_a^b} D^+F(x)\,dx \leq F(b) - F(a) \leq \overline{\int_a^b} D^+F(x)\,dx \tag{16}$$

for each interval $[a, b]$.

Proof. Take $g(x) = D^+F(x)$ in Theorem 9. ∎

5. SOME VARIANTS. We mention the following variants of Theorem 9 whose proofs can be obtained either as immediate corollaries of the theorem or by employing the same methods with slight, but familiar, adjustments.

a. The condition that F be continuous on the left at each point x can be replaced with the condition that

$$\limsup_{y \to x-} F(y) \leq F(x)$$

at each point, and inequality (10) still follows. Similarly, the condition

$$\liminf_{y \to x-} F(y) \geq F(x)$$

can replace continuity in deducing inequality (11).

b. The condition that $D^+F(x) \geq g(x)$ at every point x can be weakened to $D^+F(x) \geq g(x)$ at all but countably many points, provided that we also assume that F is continuous. (For the covering argument adjustments that need to be made in this kind of situation see [**9**].)
c. The condition that $D^+F(x) \geq g(x)$ at every point x can be weakened to $D^+F(x) \geq g(x)$ at almost every point x if we add the assumption that $D^+F(x) > -\infty$ everywhere. (Once again, the covering argument adjustments that need to be made are discussed in [**9**].)
d. If the function g lies pointwise between the two right Dini derivatives of F—that is, if

$$D_+F(x) \leq g(x) \leq \mathrm{D}^+F(x)$$

holds for each x—and if g is Lebesgue integrable (respectively, Denjoy-Perron integrable), then the conclusion of the theorem is that

$$F(b) - F(a) = \int_a^b g(x)\,dx.$$

and that F is absolutely continuous (respectively, generalized absolutely continuous in the restricted sense (see [**8**, p. 231])) on $[a, b]$.
e. If the condition that $D^+F(x) \geq g(x)$ at every point x is weakened to requiring only that

$$\limsup_{h \to 0+} \left| \frac{F(x+h) - F(x)}{h} \right| \geq g(x),$$

then the conclusion of the theorem, restated in terms of the variation of the function F, becomes

$$V(F, [a, b]) \geq \underline{\int_a^b} g(x)\,dx.$$

(The argument is exactly the same but uses inequalities of the form

$$|F(t) - F(s)| > (g(x) - \varepsilon)(t - s).)$$

6. TO RECOVER A FUNCTION FROM ONE OF ITS DINI DERIVATIVES. From Theorem 10 we have an immediate proof of the following version of Theorem 2 and, modulo certain integrability hypotheses, a complete solution of our main problem.

Theorem 11. *If F is a continuous function that has a finite Dini derivative $D^+F(x)$ at every point x of $\mathbb{R}$, then*

$$F(b) - F(a) = \int_a^b D^+F(x)\,dx$$

for each interval $[a, b]$*, provided that* $D^+F(x)$ *is integrable over* $[a, b]$ *in the sense of Riemann, of Lebesgue, or of Denjoy-Perron.*

That might well appear to be the end of the story. But Theorem 2 makes no claim that $D^+F(x)$ *must* be integrable in any of the specified senses. Indeed, such integrability cannot be inferred from the hypotheses of Theorem 10. While an ordinary derivative F' would necessarily be integrable in one of these three senses, this is not so for the Dini derivative even in the present case, where F is continuous and D^+F is everywhere finite. There is yet a *fourth* integral that intrudes here, the integral known as the Denjoy-Khintchine integral.

In the most accessible reference on this material (namely, Saks [**8**, pp. 292–293]), one finds that if a continuous function F has a finite Dini derivative $D^+F(x)$ everywhere, then that function is generalized absolutely continuous and its approximate derivative F'_{ap} is almost everywhere equal to that Dini derivative. This is exactly the setting in the Denjoy-Khintchine integration theory, in which F is an indefinite integral of D^+F. Thus, in general, the recovery of a continuous function from its everywhere finite Dini derivative seems to require this integration framework that properly encompasses all three of the Riemann, Lebesgue, and Denjoy-Perron integrals.

It is remarkable that a problem so easy to pose should lead us down such a curious path. The proof that F is determined by its Dini derivative (i.e., Theorem 1) is entirely elementary. The determination of F from that derivative, however, is far from obvious.

7. A MODIFIED INTEGRAL. The simplest version of our theory occurs if the Dini derivative $D^+F(x)$ is integrable (Riemann, Lebesgue, or Henstock-Kurzweil), for then the function F is recoverable as a limit of Riemann sums in a manner similar to what we learned as calculus students. Can we do this in general?

There is no known Riemann-sum approach available for the Denjoy-Khintchine integral. A sketch of an idea that might have worked was originally found in [**4**, Exercise 42.9, p. 222], but the author withdrew this after finding an error. The flaw was that the proposed scheme did not guarantee the existence of partitions.

But if we instead base a theory of integration on right full covers, we can solve our problem in a rather curious way. The two lemmas that are crucial to an integration theory based on right full covers are the following (the first is subsumed by Lemma 7):

Lemma 12. *If* β *is a right full cover of an interval* $[a, b]$*, then* β *contains a partition of* $[a, b]$.

Lemma 13. *If* β_1 *and* β_2 *are right full covers of an interval* $[a, b]$*, then so is the intersection* $\beta_1 \cap \beta_2$.

Proof. Let $\delta_1(x)$ and $\eta_1(x, t)$ be the functions promised in Definition 5 for β_1, and let $\delta_2(x)$ and $\eta_2(x, t)$ be the corresponding functions for β_2. If we simply take $\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$ and $\eta(x, t) = \min\{\eta_1(x, t), \eta_2(x, t)\}$, it is straightforward to verify that the statements in Definition 5 now hold for $\beta = \beta_1 \cap \beta_2$. ∎

These lemmas allow us to duplicate Definition 8 but using the more general right full covers. To indicate the new definition we use a prefix $\mathcal{RD}$ (intended to signify "right Dini") before the integral sign. We shall not trouble ourselves with an upper integral since it is not needed to state our main result.

Definition 14. Let g be a finite-valued function defined on an interval $[a, b]$. Then the *right Dini lower integral of g over* $[a, b]$ is defined by

$$(\mathcal{RD})\underline{\int_a^b} g(x)\,dx = \sup_{\beta} \inf_{\pi\subset\beta} S(g, \pi),$$

where the infimum is taken over all partitions π of $[a, b]$ contained in β and the supremum extends over all right full covers β of $[a, b]$.

We are now ready to state and prove our extension of Theorem 10 that indicates how to recover the function F from its Dini derivative, without the requirement that it be integrable.

Theorem 15. *If F is a continuous function that has a finite Dini derivative $D^+F(x)$ at every point x of $\mathbb{R}$, then*

$$\underline{\int_a^b} D^+F(x)\,dx \le (\mathcal{RD})\underline{\int_a^b} D^+F(x)\,dx = F(b) - F(a) \tag{17}$$

for each interval $[a, b]$.

Proof. The first inequality in (17), that

$$\underline{\int_a^b} D^+F(x)\,dx \le (\mathcal{RD})\underline{\int_a^b} D^+F(x)\,dx,$$

follows immediately from the fact that any full cover is also a right full cover. It remains to establish the equality in (17), which we do by the usual device of proving two inequalities. We first show that

$$(\mathcal{RD})\underline{\int_a^b} D^+F(x)\,dx \ge F(b) - F(a). \tag{18}$$

Write $g(x) = D^+F(x)$, and let $\varepsilon > 0$. Define β by equation (8), but include in β as well all pairs $(b, [s, b])$ for which

$$F(b) - F(s) - g(b)(b - s) < \varepsilon.$$

We easily check that β is a right full cover of $[a, b]$. For any partition π of $[a, b]$ contained in β we note that

$$\sum_{([s,t],x)\in\pi} g(x)(t - s) + \varepsilon(b - a) + \varepsilon \ge \sum_{([s,t],x)\in\pi} (F(t) - F(s)) = F(b) - F(a).$$

Since this is true for any partition π in β, we must have

$$F(b) - F(a) \le (\mathcal{RD})\underline{\int_a^b} g(x)\,dx + \varepsilon(b - a + 1),$$

from which inequality (18) evidently follows.

Finally, in order to complete the proof we must demonstrate that

$$(\mathcal{RD})\underline{\int_a^b} D^+F(x)\,dx \le F(b) - F(a). \tag{19}$$

Let β_1 be any right full cover of $[a, b]$, let $\varepsilon > 0$, and let β_2 be that subset of β_1 comprising those pairs $(x, [s, t])$ in β_1 $(a \le x < b)$ for which

$$\frac{F(t) - F(s)}{t - s} > g(x) - \varepsilon,$$

together with those pairs $(b, [s, b])$ in β_1 for which

$$F(b) - F(s) - g(b)(b - s) > -\varepsilon.$$

Using the continuity of F and the fact that $D^+F(x) = g(x)$ at every point x it is easily checked that β_2 (while perhaps not a right full cover) is a right *adequate* cover of $[a, b]$. Consequently, in view of Lemma 7, there is at least one partition π of $[a, b]$ contained in β_2 (hence in β_1).

For any π fitting this description we observe that

$$\sum_{([s,t],x)\in\pi} g(x)(t - s) - \varepsilon(b - a) - \varepsilon \le \sum_{([s,t],x)\in\pi} (F(t) - F(s)) = F(b) - F(a).$$

Thus, it is true that every such β_1 that is a right full cover of $[a, b]$ must contain at least one partition π with this property, so it follows that

$$F(b) - F(a) \ge (\mathcal{RD})\underline{\int_a^b} g(x)\,dx - \varepsilon(b - a + 1).$$

From this we readily infer inequality (19), completing the proof. ■

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The Continuous Adding Song (That's All Integration Is)

1.
Not just by twos, threes, fours, or fives,
We can add continu'sly.
Not just by twos, threes, fours, or fives,
We can add continu'sly.
Everybody's gladdin' bout a new way of addin'
Don't it truly blow your mind.
Not just by twos, threes, fours, or fives,
We can add continu'sly.

2.
Stretch that S, S is for sum,
It becomes an integral sign.
Stretch that S, S is for sum,
It becomes an integral sign.
Everybody's gladdin' bout a new way of addin'
That's all integration is.
Stretch that S, S is for sum,
It becomes an integral sign.

3.
Bridge the gap, from old to new,
Finite to continuous.
Go the limit, sneak up on it,
That's the role of Riemann sums.
Everybody's gladdin' bout a new way of addin'
With a limit sneak up on it.
Bridge the gap, from old to new,
That's the role of Reimann sums.

4.
If this seems just too hard to grasp,
Ponder your odometer.
It adds up the miles that you drive,
Doing so continuously.
Everybody's gladdin' bout a new way of addin'
Meters do it all the time.
If this seems just too hard to grasp,
Then ponder your odometer.

5.
F of x, then times a dx,
They're the things that we add up.
The dx is so very thin,
And x flows from left to right.
Everybody's gladdin' bout a new way of addin'
Areas of little thin strips.
F of x, then times a dx,
They're the things that we add up.

6.
Now we have a new way to add,
Aren't we rather proud of us.
No more two, three, four at a time,
We just let the summands flow.
Everybody's gladdin' bout a new way of addin'
They flow in and we add them up.
No more two, three, four at a time,
We just let the summands flow.

(Reprise and finale. Repeat if you want.)
No more two, three, four at a time,
Baby let those summands flow on.

Sung to the tune of a 1960s pop song, "Walk Right in, Sit Right down"
—Submitted by John Kiltinen, Northern Michigan University