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Review in Zentralblatt by Peter S. Bullen

The paper starts with a summary of the history of the connection of the second order symmetric derivative and the symmetric Cesro derivative with trigonometric series and the integrals introduced, using these derivatives, that turn a convergent trigonometric series into a Fourier series – the P^2-integral of James and the SCP-integral of Burkill. The authors then proceed to develop a new approach to the whole subject. This new approach is based on an idea introduced over 20 years ago by Jan Marík but never published. The orginal idea of James (and Denjoy) was to find, up to a linear function, a continuous function F from a given function f if, for all x, $a \leq x \leq b$, we have that

$$D_2 F(x) = \lim_{h \to 0} \frac{F(x+h) + F(x-h) - 2F(x)}{h^2} = f(x).$$

This F is a second order integral of f. Marík avoided the complication of a second order integral, obtaining a first order integral by determining instead

$$\Lambda_F(a,b) = \lim_{h \to 0} \frac{F(a) - F(a+h) - F(b-h) + F(b)}{h}.$$

His integral is essentially the derivative almost everywhere of the James integral. The authors follow Marík's idea but use a variational approach based on work of Henstock and of the second author. To this end they need a variation derived from the second order symmetric derivative. Thus they define

$$V_s^2(\xi, E) = \inf \Lambda_G(a,b),$$

where $\xi = \xi(x,h)$, and the infimum is taken over all G, convex functions on an open interval containing $\bar{E}$, E a bounded set, and such that for each $x \in E$ there is a $\delta = \delta(x,G)$ and for all $h, 0 < h < \delta$, $|\xi(x,h)| \leq \Delta_s^2 G(x,h) = G(x+h) + G(x-h) - 2G(x)$. As a function of E, V_s^2 is an outer measure and we say that $\xi_1 \equiv \xi_2$, the two functions are variationally equivalent, if $V_2^s(\xi_1 - \xi_2, E) = 0$. The theory required is obtained by taking ξ as either $\Delta_s^2 F(x)$ or $\Delta_s^2 F(x) - f(x)h^2$; then f is integrable if $\Delta_s^2 \equiv f(x)h^2$ for some continuous F; this follows ideas used in the theory of the Henstock integral.

The advantages of their approach become clear by the manner in which the often awkward relations and properties of the classical integrals are illuminated. They also obtain a simplification of Skljarenko's proof of the integration by parts theorem for the SCP-integral. Finally, they show that their integral will turn a convergent trigonometric series into a Fourier series.

The paper is written with exceptional clarity and is not only the last word in the subject of the title but also an essential introduction to that subject as well. [P.S.Bullen (Vancouver)]