THE BIPARTITE BOOLEAN QUADRIC POLYTOPE

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Abstract. We consider the *Bipartite Boolean Quadratic Programming Problem* (BQP01), which generalizes the well-known Boolean quadratic programming problem (QP01). The model has applications in graph theory, matrix factorization and bioinformatics, among others. The primary focus of this paper is on studying the structure of the *Bipartite Boolean Quadric Polytope* ($\text{BQP}^{m,n}$) resulting from a linearization of a quadratic integer programming formulation of BQP01.

We present some basic properties and partial relaxations of $\text{BQP}^{m,n}$, as well as some families of facets and valid inequalities. We find facet-defining inequalities including a family of odd-cycle inequalities. We discuss various approaches to obtain a valid inequality and facets from those of the related Boolean quadric polytope. The key strategy is based on rounding coefficients, and it is applied to the families of clique and cut inequalities in $\text{BQP}^{m,n}$.

1. Introduction

In this paper, we investigate a special quadratic programming problem where feasible solutions are extreme points of the unit cube in $\mathbb{R}^{m+n}$. Let $Q = (q_{i,j})$ be an $m \times n$ matrix where $m \leq n$, $c = (c_1, \dots, c_m)$ be a row vector in $\mathbb{R}^m$, $d = (d_1, \dots, d_n)$ be a row vector in $\mathbb{R}^n$ and c_0 be a constant. Then the *Bipartite Boolean Quadratic Programming Problem* (BQP01) is

$$\text{Maximize } f(x,y) = x^T Q y + cx + dy + c_0$$
$$\text{Subject to } x \in \{0,1\}^m, \quad y \in \{0,1\}^n.$$

A graph theoretic interpretation of BQP01 can be given as follows. Let $G(I, J, E)$ be a complete bipartite graph, where $I = \{1, \dots, m\}$ and $J = \{1, \dots, n\}$. Let $q_{i,j}$ be the weight of the edge (i,j) where $i \in I$ and $j \in J$, c_i be the weight of vertex $i \in I$ and d_j be the weight of vertex $j \in J$. Let $S(I', J', E')$ be a subgraph of G. The weight of S is the total weight of its vertices and edges. Selecting a maximum weight complete bipartite subgraph of G is precisely the BQP01.

Throughout this paper, we denote (i,j) (or (j,i)) the edge between vertices i and j. Moreover, if (i,j) is an edge in a bipartite graph $G(I,J,E)$, the first entry i stands for the vertex in I and the second entry j stands for the vertex in J.

The model BQP01 has many applications. Consider a bipartite graph $G(I,J,E)$. A subgraph $S'(I', J', E')$ of G is said to be a biclique if S' is a complete bipartite graph. By the graph theoretic interpretation mentioned above, *Maximum Biclique Problem* (MBCP) and *Maximum Weighted Biclique Problem* (MWBCP) in G can be solved as BQP01. The problem MBCP and MWBCP have been studied by various authors. Applications of these models include data mining [19], clustering [6, 21] and bioinformatics [21, 27].

Approximating matrices using low-rank $\{0,1\}$-matrices can be modelled as a BQP01 problem. It has applications in mining discrete patterns in binary data [28] and various clustering applications. Many authors considered the problem of approximating a matrix by a rank-one binary matrix [10, 16, 17, 28]. Other application areas of BQP01 include finding the cut-norm of a matrix [1], correlation clustering [5, 32], bioinformatics [5, 32] and approximating matrices using $\{-1,1\}$ entries [28].

Besides MWBCP, another problem closely related to BQP01 is *Boolean Quadratic Programming Problem* (QP01), a well studied combinatorial optimization problem with various applications [3]. The problem

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can be stated as

$$\begin{aligned}&\text{Maximize } u^T Q' u + c' u + c'_0\\&\text{Subject to } u \in \{0,1\}^n,\end{aligned}$$

where $Q' = (q'_{i,j})$ is an $n \times n$ matrix, c' is a row vector of length n, u is a column vector of size n and c'_0 is a constant. BQP01 can be viewed as a special case of QP01 in a higher dimension and QP01 can be formulated as a BQP01 problem as well [25]. The QP01 problem is also known as Binary Quadratic Programming and abbreviated BQP. To avoid confusion with the bipartite BQP01 case, we exclusively use the abbreviation QP01 for the Boolean Quadratic Programming Problem, and BQP01 for the Bipartite Boolean Quadratic Programming Problem.

Exact algorithms for QP01 are mostly enumerative, involving branch-and-bound or branch-and-cut methods. Many of these algorithms use bounds generated by various relaxations including several linearization methods. One of the most well-studied linearization of QP01 was suggested by Padberg [23]. In this work, QP01 is formulated as an integer linear programming problem by introducing additional constraints and variables $v \in \{0,1\}^{(n^2-n)/2}$ where $v_{i,j} = u_i u_j$ as

$$\text{Maximize } \sum_{i<j} q'_{i,j} v_{i,j} + c'u + c_0$$

$$\text{Subject to } \quad u_i + u_j - v_{i,j} \le 1, \tag{1.1}$$

$$-u_i \quad + v_{i,j} \le 0, \tag{1.2}$$

$$-u_j + v_{i,j} \le 0, \tag{1.3}$$

$$-v_{i,j} \le 0, \tag{1.4}$$

$$u_i, v_{i,j} \text{ integer, for all } 1 \le i < j \le n. \tag{1.5}$$

Padberg studied the convex hull QP^n of all feasible solutions of (1.1) to (1.5), which is called the *Boolean Quadric Polytope*. Four families of facets of the polytope QP^n were studied by Padberg [23], which include: trivial facets, clique inequalities, cut inequalities, and generalized cut inequalities. In general, these four families of facets are not sufficient to describe all the facets of QP^n. Sherali et al. [30] applied a simultaneous lifting procedure to obtain an additional family of facet-defining inequalities for QP^n. Macambira and Souza [20] studied the edge-weighted clique problem and found that some facet-defining inequalities for the polytope corresponding to this problem also define facets for QP^n.

Similarly, we are interested in a linearization of BQP01 and its corresponding polytope. Linearization of BQP01 can be achieved by introducing a new variable $z_{i,j} = x_i y_j$ which has zero-one value and satisfies

$$x_i + y_j - z_{i,j} \le 1, \tag{1.6}$$

$$-x_i \quad + z_{i,j} \le 0, \tag{1.7}$$

$$-y_j + z_{i,j} \le 0, \tag{1.8}$$

$$-z_{i,j} \le 0, \tag{1.9}$$

$$x_i, y_j, z_{i,j} \text{ integer,} \tag{1.10}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

This type of linearization has a long history, see for example [13, 12, 22, 29, 11], and continues to be fruitful, see for example [34, 7, 33, 31, 35].

We denote the convex hull of the solutions of (1.6)–(1.10)

$$\mathrm{BQP}^{m,n} = \mathrm{conv}\left\{(x,y,z) \in \mathbb{R}^{m+n+mn} : (x,y,z) \text{ satisfies (1.6)–(1.10)}\right\}$$

the *Bipartite Boolean Quadric Polytope* and

$$\mathrm{BQP}^{m,n}_{LP} = \left\{(x,y,z) \in \mathbb{R}^{m+n+mn} : (x,y,z) \text{ satisfies (1.6)–(1.9)}\right\}$$

the linear relaxation of $\mathrm{BQP}^{m,n}$. Both are relevant in developing algorithms for various applications of the BQP01 model. Several recent papers have explored heuristics for the bipartite Boolean quadratic programming problem [9, 14, 15, 24, 36].

In this paper, we present some basic properties and partial relaxations of the polytope as well as some families of facets and valid inequalities. We provide trivial inequalities and lifts of I_{mm22} Bell inequalities which define facets of the polytope. Odd-cycle inequalities are a family of valid inequalities which are facet-defining under a condition on the underlying cycle. Also, we use a coefficient rounding technique to obtain a valid inequality from a facet-defining inequality of the polytope corresponding to QP01. Similar techniques have recently been applied to the Bipartite Boolean Quadric Polytope with additional multiple choice constraints [4].

The results in this paper are included in the Ph.D. thesis of the first author [26].

2. Basic Properties of $\mathrm{BQP}^{m,n}$

Padberg [23] gave many results on the Boolean quadric polytope QP^n. We consider similar properties for $\mathrm{BQP}^{m,n}$. It is easy to see that the following two facts, known for QP^n (see e.g. [23]), are also true for $\mathrm{BQP}^{m,n}$ and $\mathrm{BQP}^{m,n}_{LP}$.

Proposition 2.1. *If $(x, y, z) \in \mathrm{BQP}^{m,n}$ or $(x, y, z) \in \mathrm{BQP}^{m,n}_{LP}$, then $0 \le x_i \le 1$, $0 \le y_j \le 1$ and $0 \le z_{i,j} \le 1$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. It follows that $\mathrm{BQP}^{m,n} \subseteq [0, 1]^{m+n+mn}$ and $\mathrm{BQP}^{m,n}_{LP} \subseteq [0, 1]^{m+n+mn}$.*

Since all 2^{m+n} $\{0, 1\}$-points satisfying (1.6)–(1.10) are solutions of BQP01, this proposition implies that these 2^{m+n} points are vertices of $\mathrm{BQP}^{m,n}$. Since $\mathrm{BQP}^{m,n}_{LP} \subseteq [0, 1]^{m+n+mn}$, these 2^{m+n} $\{0, 1\}$-points are also vertices of $\mathrm{BQP}^{m,n}_{LP}$.

We denote by u^i the vector in $\mathbb{R}^m$ with all zero entries except the i^{th} which is equal to 1, by v^j the vector in $\mathbb{R}^n$ with all zero entries except the j^{th} which is equal to 1 and by $w^{i,j}$ the vector in $\mathbb{R}^{mn}$ with all zero entries except the i, j entry which is equal to 1. Moreover, we denote 0^k the zero vector of length k.

Proposition 2.2. *$\mathrm{BQP}^{m,n}$ and $\mathrm{BQP}^{m,n}_{LP}$ are full-dimensional, that is, $\dim(\mathrm{BQP}^{m,n}) = \dim(\mathrm{BQP}^{m,n}_{LP}) = m + n + mn$.*

Proof. Consider $m + n + mn + 1$ points in $\mathrm{BQP}^{m,n}$, namely $0^{m+n+mn}, (u^i, 0^n, 0^{mn})$ for $i = 1, \ldots, m$, $(0^m, v^j, 0^{mn})$ for $j = 1, \ldots, n$ and $(u^i, v^j, w^{i,j})$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. These are $m+n+mn+1$ affinely independent points in $\mathrm{BQP}^{m,n}$. Therefore, the polytope has dimension $m+n+mn$ and it is full-dimensional. Since $\mathrm{BQP}^{m,n} \subseteq \mathrm{BQP}^{m,n}_{LP}$, it follows that $\mathrm{BQP}^{m,n}_{LP}$ is also full-dimensional and its dimension is $m + n + mn$. □

2.1. Partial Linear Relaxation of $\mathrm{BQP}^{m,n}$.

When we define the linear relaxation of $\mathrm{BQP}^{m,n}$, we allow all entries to be non-integral. In this section, we consider the case when we keep the integral constraints on only one of x, y or z.

Denote $\mathrm{BQP}^{m,n}_x$ the convex hull of points $(x, y, z) \in \mathbb{R}^{m+n+mn}$ satisfying (1.6)–(1.9) and $x_i \in \{0, 1\}$ for $i = 1, \ldots, m$; $\mathrm{BQP}^{m,n}_y$ the convex hull of points $(x, y, z) \in \mathbb{R}^{m+n+mn}$ satisfying (1.6)–(1.9) and $y_j \in \{0, 1\}$ for $j = 1, \ldots, n$ and $\mathrm{BQP}^{m,n}_z$ the convex hull of points $(x, y, z) \in \mathbb{R}^{m+n+mn}$ satisfying (1.6)–(1.9) and $z_{i,j} \in \{0, 1\}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

All properties of $\mathrm{BQP}^{m,n}$ shown earlier in this work are also true for $\mathrm{BQP}^{m,n}_x$, $\mathrm{BQP}^{m,n}_y$ and $\mathrm{BQP}^{m,n}_z$ with the same proofs, that is, these three polytopes are bounded in $[0, 1]^{m+n+mn}$ and full-dimensional. In fact, we can show that $\mathrm{BQP}^{m,n}_x$=$\mathrm{BQP}^{m,n}_y$=$\mathrm{BQP}^{m,n}_z$=$\mathrm{BQP}^{m,n}$.

Theorem 2.3. $\mathrm{BQP}_x^{m,n}$=$\mathrm{BQP}_y^{m,n}$=$\mathrm{BQP}^{m,n}$.

Proof. By symmetry, it suffices to prove that $\mathrm{BQP}_x^{m,n}$=$\mathrm{BQP}^{m,n}$. Consider the polyhedron with the constraints

$$\begin{aligned} x_i + y_j - z_{i,j} &\le 1, \\ -x_i \qquad + z_{i,j} &\le 0, \\ -y_j + z_{i,j} &\le 0, \\ x_i, y_j, z_{i,j} &\ge 0, \end{aligned}$$

for $i = 1, \dots, m$ and $j = 1, \dots, n$. We can fix x and treat each x_i as a constant in $\{0, 1\}$. Then we can rewrite them as

$$\begin{aligned} y_j - z_{i,j} &\le 1 - x_i, \\ z_{i,j} &\le x_i, \\ -y_j + z_{i,j} &\le 0, \\ y_j, z_{i,j} &\ge 0, \end{aligned}$$

for $i = 1, \dots, m$ and $j = 1, \dots, n$. It is known that $P_x := \{(y, z) : A(y, z) \le b, y, z \ge 0\}$ is integral if b is integral and the constraint matrix A is totally unimodular (Theorem 2 of [18]). The constraint matrix A for our constraints is

$$A^T = \left[\begin{array}{c|c|c} I_n \cdots I_n & -I_n \cdots - I_n & 0_{n\times mn} \\ \hline -I_{mn} & I_{mn} & I_{mn} \end{array}\right],$$

where I_k is the identity matrix of dimension $k \times k$ and $0_{h\times k}$ is the zero matrix with h rows and k columns, where h and k are positive integers. The matrix has dimension $3mn \times (n + mn)$. We claim that this constraint matrix is totally unimodular.

We can see that each entry of $A = (a_{i,j})$ is in $\{-1, 0, 1\}$. Moreover, each row of A has at most two nonzero entries. Let C be the set of all columns of A. Consider C as a disjoint union of C and $\emptyset$. For each row $i = 1, 2, \dots, 2mn$ with exactly two entries, there are one 1 and one -1. Hence, $\sum_{j\in C} a_{i,j} = 0 = \sum_{j\in\emptyset} a_{ij}$. By Theorem 5 of [18], A is totally unimodular. Note that $\mathrm{BQP}_x^{m,n}$ is the convex hull of a finite set of polytopes obtained by fixing x to an integer vector. We have shown above that every such polytope is integer. Thus, $\mathrm{BQP}_x^{m,n}$ is integer as well. Therefore, $\mathrm{BQP}_x^{m,n}$ is also integral. Since $\mathrm{BQP}_x^{m,n}$ is a relaxation of $\mathrm{BQP}^{m,n}$, we have $\mathrm{BQP}^{m,n} \subseteq \mathrm{BQP}_x^{m,n}$. Moreover, the fact that every vertex of $\mathrm{BQP}_x^{m,n}$ satisfies integrality constraints on y and z implies that $\mathrm{BQP}_x^{m,n} \subseteq \mathrm{BQP}^{m,n}$. Hence, we can conclude that $\mathrm{BQP}_x^{m,n}$ =$\mathrm{BQP}^{m,n}$. $\square$

We can use this technique to show that $\mathrm{BQP}_z^{m,n}$ =$\mathrm{BQP}^{m,n}$.

Theorem 2.4. $\mathrm{BQP}_z^{m,n}$=$\mathrm{BQP}^{m,n}$.

Proof. We consider the same polyhedron as in the previous proof. Fix $z \in \{0, 1\}^{mn}$ and see each $z_{i,j}$ as a constant in $\{0, 1\}$. Hence, we can restate the constraints as

$$\begin{aligned} x_i + y_j &\le 1 + z_{i,j}, \\ -x_i \qquad &\le -z_{i,j}, \\ - y_j &\le -z_{i,j}, \\ x_i, y_j &\ge 0, \end{aligned}$$

for $i = 1, \dots, m$ and $j = 1, \dots, n$. Similar to the proof of the previous theorem, we use the fact that $P_z := \{(x, y) : A(x, y) \le b, x, y \ge 0\}$ is integral if b is integral and the constraint matrix A is totally unimodular. Using notation u_i as defined earlier in this section, the constraint matrix A is given by

$$A^T = \left[\begin{array}{c|c|c|c|c} u_1 \cdots u_1 & \cdots & u_m \cdots u_m & -I_m \cdots - I_m & 0_{m\times mn} \\ \hline I_n & \cdots & I_n & 0_{n\times mn} & -I_n \cdots - I_n \end{array}\right],$$

which has dimension $3mn \times (m + n)$.

Note that each entry of $A = (a_{i,j})$ is in $\{-1, 0, 1\}$. Also, each row of A has at most two entries. Let C be the set of all columns of A. Partition C into $\{1, \ldots, m\}$ and $\{m+1, \ldots, m+n\}$. For each row $i = 1, 2, \ldots, mn$ with exactly two entries, $\sum_{i=1}^{m} a_{i,j} = 1 = \sum_{i=m+1}^{m+n} a_{i,j}$. We can conclude that A is totally unimodular, using Theorem 5 of [18]. Similar to the previous proof, we have $\mathrm{BQP}_z^{m,n} = \mathrm{BQP}^{m,n}$. □

2.2. **Properties as a Restriction of the Boolean Quadric Polytope.** In this section we deduce some properties of BQP01 by viewing it as a restriction of higher dimensional instance of QP01.

Definition 2.5. [23] Let Q be an upper triangular matrix of size $n \times n$ with zero diagonal. Then we can define $G(V, E)$ as a graph on n vertices spanned by the edges $e = (i, j)$ given by nonzero coefficients $q_{i,j}$ of Q. Note that G has no loops because all diagonal entries of Q are zeros. Without loss of generality, we can assume that G has no isolated vertices. We denote

$$\mathrm{QP}^G = \mathrm{conv}\left\{(u, v) \in \mathbb{R}^{|V|+|E|} : (u, v) \text{ satisfies } (1.1), \ldots, (1.5) \text{ for all } (i, j) \in E\right\}$$

the *Boolean Quadric Polytope associated with* G.

From this definition, we can see QP^n as QP^G where $G = K_n$. Thus, $\mathrm{BQP}^{m,n}$ can be viewed as a Boolean quadric polytope associated with a biclique whose partite sets have order m and n, that is $\mathrm{BQP}^{m,n}=\mathrm{QP}^{K_{m,n}}$. Therefore, we can apply some properties of the Boolean quadric polytope provided by Padberg [23] to QP^G where G is bipartite.

Since QP^G has only trivial facets if and only if G is acyclic [23] and $K_{m,n}$ is acyclic if and only if $m = 1$ or $n = 1$, we obtain the following result for $\mathrm{BQP}^{m,n}$.

Proposition 2.6. $\mathrm{BQP}^{m,n} = \mathrm{BQP}_{LP}^{m,n}$ *if and only if* m *or* n *is* 1. *Consequently, all the facets of* $\mathrm{BQP}^{m,n}$ *are trivial if and only if* m *or* n *is* 1.

Next, we introduce some definitions and notations, based on [23], before showing other results obtained from Padberg's work.

Let $a = (a^1, a^2, a^3)$ be a row vector of size $m + n + mn$ where a^1 is a row vector of size m, a^2 is a row vector of size n and a^3 is a row vector of size mn. For any valid inequality $a\omega := a^1x + a^2y + a^3z \le a_0$ of $\mathrm{BQP}^{m,n}$, we can define its support graph.

Definition 2.7. For any valid inequality $a\omega \le a_0$ of $\mathrm{BQP}^{m,n}$, the *support graph* of the inequality is denoted by $G(a) = (V_a, E_a)$ where $E_a = \left\{(i, j) \in E : a^3_{i,j} \ne 0\right\}$ and V_a is the subset of V spanned by E_a.

Note that $G(a)$ is bipartite. For a facet-defining inequality $\alpha\tau \le \alpha_0$ of QP^n, Padberg [23] gave an interesting property on vector α and the support graph $G(\alpha)$. We give a $\mathrm{BQP}^{m,n}$ version of this property. The proof is very similar to that of QP^n version, and hence, is omitted.

Proposition 2.8. *If* $a\omega := a^1x + a^2y + a^3z \le a_0$ *defines a facet of* $\mathrm{BQP}^{m,n}$, *there is a pair* (i, j) *such that* $a^3_{i,j} \ne 0$.

In general, when $H = (V_H, E_H)$ is a subgraph of G, we can talk about an *extension* of a valid inequality $\alpha^H\tau \le \alpha_0^H$ for QP^H to QP^G obtained from $\alpha^H\tau \le \alpha_0^H$ by adding components $\alpha_i^G = 0$ for any $i \in V \backslash V_H$ and $\alpha_{i,j}^G = 0$ where $(i, j) \in E\backslash E_H$. The following proposition shows a relationship of facet-defining inequalities of two Boolean quadric polytopes associated to different graphs G and H where $H \subseteq G$ when G and H are bipartite.

Proposition 2.9. [23] *If* $a\omega \le a_0$ *defines a facet for* QP^H, *its extension defines a facet for* QP^G *for any bipartite* G *containing* H *as an induced subgraph.*

We can easily see that if H is a biclique, this proposition applies to any bipartite G containing H. Note that $\mathrm{BQP}^{m,n}$ is QP^H where $H = K_{m,n}$. Let $M \ge m$ and $N \ge n$, then $\mathrm{BQP}^{M,N} = \mathrm{QP}^G$ where $G = K_{M,N}$. It follows that $K_{M,N}$ contains $H = K_{m,n}$ as an induced subgraph. Hence, the extension of a facet-defining inequality for $\mathrm{BQP}^{m,n}$ also defines a facet for $\mathrm{BQP}^{M,N}$.

Let $a\omega := a^1x + a^2y + a^3z \le a_0$ be a valid inequality for $\mathrm{BQP}^{m,n}$. The *canonical extension* $\hat{a}\omega \le a_0$ of $a\omega \le a_0$ to $\mathrm{BQP}^{M,N}$ for $M \ge m$ and $N \ge n$ is

$$\hat{a}_i^1 = \begin{cases} a_i^1 & \forall i \in I, \\ 0 & \forall i \in I' \backslash I \end{cases}, \quad \hat{a}_j^2 = \begin{cases} a_j^2 & \forall j \in J, \\ 0 & \forall j \in J' \backslash J, \end{cases} \text{ and } \hat{a}_{i,j}^3 = \begin{cases} a_{i,j}^3 & \forall i \in I, j \in J, \\ 0 & \text{otherwise,} \end{cases}$$

where $I = \{1, \ldots, m\}, J = \{1, \ldots, n\}, I' = \{1, \ldots, M\}$, and $J' = \{1, \ldots, N\}$. Then we obtain the following corollary.

Corollary 2.10. *If $a\omega := a^1x + a^2y + a^3z \le a_0$ defines a facet of $\mathrm{BQP}^{m,n}$, then the canonical extension $\hat{a}\omega \le a_0$ defines a facet of $\mathrm{BQP}^{M,N}$ for $M \ge m$ and $N \ge n$.*

Now, we can give a new family of facet-defining inequalities for $\mathrm{BQP}^{m,n}$. The I_{mm22} *Bell inequalities* are the inequalities in the form

$$-x_1 - \sum_{j=1}^{m}(m-j)y_j - \sum_{2 \le i,j \le m, i+j=m+2} z_{i,j} + \sum_{1 \le i,j \le m, i+j \le m+1} z_{i,j} \le 0, \tag{2.1}$$

for $m \ge 1$. Avis and Ito [2] gave a proof that this inequality is facet-defining for $\mathrm{BQP}^{m,m}$. Applying Corollary 2.10, we get the following result.

Corollary 2.11. *The canonical extension of an inequality* (2.1) *for $\mathrm{BQP}^{l,l}$ to $\mathrm{BQP}^{m,n}$, for some $l \le m$, defines a facet for $\mathrm{BQP}^{m,n}$.*

For $l = 1$, the canonical extension of the I_{1122} Bell inequalities from $\mathrm{BQP}^{1,1}$ to $\mathrm{BQP}^{m,n}$ gives the trivial facet-defining inequality (1.7). When $l = 2$, the canonical extension of the I_{2222} Bell inequalities are of the form $-x_h - y_j - z_{i,k} + z_{h,k} + z_{i,j} + z_{h,j} \le 0$, where $h, i \in I$ and $j, k \in J$, or vice versa. By the symmetry of x_h and y_j, each inequality is determined by the pair of vertices from different partite sets to be x_h and y_j, one vertex from the same partite set as x_h plays the role of x_i, and one vertex from the same partite set as y_j plays the role of y_k. There are mn ways to choose a pair of x_h and y_j, and then $(m-1)(n-1)$ to choose x_i and y_k. Therefore, we have $mn(m-1)(n-1)$ facets obtained from I_{2222} Bell inequalities.

For any integer $3 \le l \le m$, each canonical extension of I_{ll22} Bell inequality corresponds to one labelled $K_{1,l}(I', J')$ subgraph. If the vertex in I' is chosen from I, there are m ways to choose a vertex in I', $\binom{n}{l}$ ways to choose l vertices in J to form J', $l!$ ways to label these l vertices, and hence, $m\binom{n}{l}l!$ ways in total. Similarly, if the vertex in I' is chosen from J, there are $n\binom{m}{l}l!$ ways to construct a canonical extension of I_{ll22} Bell inequality. Thus, we get $mn(m-1)(n-1) + \sum_{l=3}^{m}(n\binom{m}{l} + m\binom{n}{l})l!$ non-trivial facets from this method.

3. Families of Facet-defining Inequalities and Valid Inequalities of the Polytope

It is challenging to enumerate the facets of QP^G even for very small G, see for example [8]. The properties of QP^G where G is an acyclic graph or a series-parallel graph were studied by Padberg [23]. Otherwise, finding facets of QP^G when G is not a complete graph is a hard problem that is less studied compared to QP^n. In this section, we develop various families of facet-defining inequalities and valid inequalities for QP^G, emphasizing the case $G = K_{m,n}$.

3.1. **Trivial Facets.** We show that all inequalities in the form (1.6)-(1.9) define facets of $\mathrm{BQP}^{m,n}$ and $\mathrm{BQP}^{m,n}_{LP}$. We call these facets *trivial facets* of $\mathrm{BQP}^{m,n}$. To prove that an equality defines a facet, we show that the hyperplane corresponding to the inequality contains $m + n + mn$ affinely independent points in the polytope, which implies that the facet has dimension $m + n + mn - 1$. Here, we use the notation $u^i, v^j, w^{i,j}$ and 0^k as introduced in the preamble to Proposition 2.2: recall that u^i and v^j have unique non-zero entries corresponding to vertices i and j on their respective sides of the bipartition, while $w^{i,j}$ has non-zero entries corresponding to vertices i and j as well as edge ij.

Lemma 3.1. *The inequalities $-z_{i,j} \le 0$ define facets of $\mathrm{BQP}^{m,n}$ and $\mathrm{BQP}^{m,n}_{LP}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.*

Proof. Let $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Define $F = \{(x, y, z) \in \mathrm{BQP}^{m,n} : z_{i,j} = 0\}$. We consider

- $(0^m, 0^n, 0^{mn})$;
- $(u^h, 0^n, 0^{mn})$ for $h = 1, \ldots, m$;
- $(0^m, v^k, 0^{mn})$ for $k = 1, \ldots, n$ and
- $(u^h, v^k, w^{h,k})$ for $h = 1, \ldots, m$, $k = 1, \ldots, n$ and $(h, k) \neq (i, j)$.

These are $m + n + mn$ points satisfying (1.6)–(1.10) and $z_{i,j} = 0$ and it is routine to check that they are affinely independent. Therefore, $-z_{i,j} \leq 0$ defines a facet of $\mathrm{BQP}^{m,n}$. Note that these $m + n + mn$ points are also in $\mathrm{BQP}^{m,n}_{LP}$. Hence, $-z_{i,j} \leq 0$ also defines a facet of $\mathrm{BQP}^{m,n}_{LP}$. □

Lemma 3.2. *The inequalities $-x_i + z_{i,j} \leq 0$ and $-y_j + z_{i,j} \leq 0$ define facets of $\mathrm{BQP}^{m,n}$ and $\mathrm{BQP}^{m,n}_{LP}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.*

Proof. We will show only the proof for $-x_i + z_{i,j} \leq 0$. Similar arguments can be applied for $-y_j + z_{i,j} \leq 0$. Let $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Define $F = \{(x, y, z) \in \mathrm{BQP}^{m,n} : -x_i + z_{i,j} = 0\}$. We consider

- $(0^m, 0^n, 0^{mn})$;
- $(u^h, 0^n, 0^{mn})$ for $h = 1, \ldots, m$ where $h \neq i$;
- $(0^m, v^k, 0^{mn})$ for $k = 1, \ldots, n$;
- $(u^i, v^j, w^{i,j})$;
- $(u^h, v^k, w^{h,k})$ such that $h = 1, \ldots, m$, $h \neq i$ and $k = 1, \ldots, n$ and
- $(u^i, v^j + v^k, w^{i,j} + w^{i,k})$ for $k = 1, \ldots, n$ and $k \neq j$.

These $m + n + mn$ points are in F and are affinely independent. Therefore, $-x_i + z_{i,j} \leq 0$ defines a facet of $\mathrm{BQP}^{m,n}$. Since these $m + n + mn$ points are also in $\mathrm{BQP}^{m,n}_{LP}$, $-x_i + z_{i,j} \leq 0$ also defines a facet of $\mathrm{BQP}^{m,n}_{LP}$. □

Lemma 3.3. *The inequalities $x_i + y_j - z_{i,j} \leq 1$ define facets of $\mathrm{BQP}^{m,n}$ and $\mathrm{BQP}^{m,n}_{LP}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.*

Proof. Let $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Fix some $i' \in \{1, \ldots, m\}$ and $j' \in \{1, \ldots, n\}$ such that $i' \neq i$ and $j' \neq j$. Define $F = \{(x, y, z) \in \mathrm{BQP}^{m,n} : x_i + y_j - z_{i,j} = 1\}$. We consider

- $(u^i, 0^n, 0^{mn})$;
- $(0^m, v^j, 0^{mn})$;
- $(u^i, v^j, w^{i,j})$;
- $(u^i + u^h, v^j + v^k, w^{i,j} + w^{i,k} + w^{h,j} + w^{h,k})$ such that $h = 1, \ldots, m$, $h \neq i$ and $k = 1, \ldots, n$, $k \neq j$;
- $(u^h, v^j, w^{h,j})$ for $h = 1, \ldots, m$, $h \neq i$;
- $(u^i, v^k, w^{i,k})$ for $k = 1, \ldots, n$, $k \neq j$;
- $(u^i + u^h, v^{j'}, w^{i,j'} + w^{h,j'})$ for $h = 1, \ldots, m$, $h \neq i$ and
- $(u^{i'}, v^j + v^k, w^{i',j} + w^{i',k})$ for $k = 1, \ldots, n$, $k \neq j$.

It is not difficult to verify that these $m + n + mn$ points are in F and are affinely independent. These $m + n + mn$ points are also in $\mathrm{BQP}^{m,n}_{LP}$. Therefore, $x_i + y_j - z_{i,j} \leq 1$ defines a facet of $\mathrm{BQP}^{m,n}$ and $\mathrm{BQP}^{m,n}_{LP}$. □

These lemmas give us $4mn$ facets in total, mn facets for each type of inequality. The results are summarized in the next Theorem.

Theorem 3.4. (i) $\dim(\mathrm{BQP}^{m,n}) = \dim(\mathrm{BQP}^{m,n}_{LP}) = m + n + mn$
(ii) *The inequalities* (1.6)-(1.9) *define the facets of* $\mathrm{BQP}^{m,n}_{LP}$
(iii) *The inequalities* (1.6)-(1.9) *define trivial facets of* $\mathrm{BQP}^{m,n}$

We remark that this means the description of $\mathrm{BQP}^{m,n}_{LP}$ via (1.6)–(1.9) is complete and irredundant.

3.2. **Odd-Cycle Inequalities.** Padberg [23] introduced the family of odd-cycle inequalities for QP^n. These odd-cycle inequalities for QP^n are derived from triangle inequalities, a family of valid inequalities for QP^n. Since there are no triangle inequalities for $\mathrm{BQP}^{m,n}$, we begin by verifying the validity of odd-cycle inequalities for $\mathrm{BQP}^{m,n}$, and more generally QP^G, which are not elaborated in [23].

Let $G(V,E)$ be a graph and $C(V_C,E_C)$ be any simple cycle of length at least 3 in G. For any subset M of E_C where $|M|$ is odd, define

$$S_M = \{w \in V_C : \exists\ (u,w) \neq (v,w) \in M\}\text{, and}$$
$$S'_M = \{w \in V_C : \exists\ (u,w) \neq (v,w) \in E_C \backslash M\}.$$

For $S \subseteq V$, we define $\tau^S = (u^S, v^S) \in \mathrm{QP}^G$, the *characteristic vector of* S, by

$$u_i^S = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \in V\backslash S, \end{cases} \qquad v_{i,j}^S = \begin{cases} 1 & \text{if } (i,j) \in E(S), \\ 0 & \text{if } (i,j) \in E\backslash E(S), \end{cases}$$

where $E(S) = \{(i,j) : i,j \in S\}$. We can see that $u^S \in \{0,1\}^{|V|}$ and $v^S \in \{0,1\}^{|E|}$. Moreover, for any $(u,v) \in \mathrm{QP}^G$ where $u \in \{0,1\}^{|V|}$ and $v \in \{0,1\}^{|E|}$, (u,v) is a characteristic vector for some $S \subseteq V$.

Let $(u,v) \in \mathrm{QP}^G$ where $u \in \{0,1\}^{|V|}$ and $v \in \{0,1\}^{|E|}$. For any variable ξ_i whose index corresponds to a vertex in V, denote $\xi(S) = \sum_{i \in S} \xi_i$. Let $F \subseteq E$. For any variable $\zeta_{i,j}$ whose index corresponds to an edge in E, denote $\zeta(F) = \sum_{(i,j)\in F} \zeta_{i,j}$. An odd-cycle inequality for QP^G is an inequality in the form

$$u(S_M) - u(S'_M) + v(E_C\backslash M) - v(M) \leq \left\lfloor \frac{|M|}{2} \right\rfloor. \tag{3.1}$$

Proposition 3.5. [23] *An odd-cycle inequality is a valid inequality for* QP^G.

For $\mathrm{BQP}^{m,n}$, we can define a characteristic vector in a similar way. Let $G(V,E)$ be the underlying graph of the polytope. Then G is a complete bipartite graph whose partite sets I and J contain m and n vertices, respectively. Let $S \subseteq V$. We define $\omega^S = (x^S, y^S, z^S) \in \mathrm{BQP}^{m,n}$, the *characteristic vector of* S, by

$$x_i^S = \begin{cases} 1 & \text{if } i \in S \cap I, \\ 0 & \text{if } i \in I\backslash S, \end{cases} \qquad y_j^S = \begin{cases} 1 & \text{if } j \in S \cap J, \\ 0 & \text{if } j \in J\backslash S, \end{cases} \qquad z_{i,j}^S = \begin{cases} 1 & \text{if } (i,j) \in (S\cap I : S \cap J), \\ 0 & \text{if } (i,j) \in E\backslash(S\cap I : S\cap J), \end{cases}$$

where $(S_1 : S_2) = \{(i,j) : i \in S_1, j \in S_2\}$ for any disjoint subsets S_1 and S_2 of V. We can see that any vector (x,y,z) satisfying $(1.6), \ldots, (1.10)$ is a characteristic vector for some subset S of V. We can write an odd-cycle inequality for $\mathrm{BQP}^{m,n}$ as

$$x(S_M) - x(S'_M) + y(S_M) - y(S'_M) + z(E_C\backslash M) - z(M) \leq \left\lfloor \frac{|M|}{2} \right\rfloor. \tag{3.2}$$

Note that when G is bipartite, C is an even cycle. Thus, we have $|C| \geq 4$. Since $\mathrm{BQP}^{m,n} = \mathrm{QP}^{K_{m,n}}$, we get the following corollary.

Corollary 3.6. *An odd-cycle inequality is a valid inequality for* $\mathrm{BQP}^{m,n}$.

Since each odd-cycle inequality is determined by the choice of cycle C and the set M, there are $\sum_{i=2}^{m} \binom{m}{i}\binom{n}{i}(\sum_{j=1}^{i}\binom{2i}{2j-1})$ odd-cycle inequalities for $\mathrm{BQP}^{m,n}$.

The following proposition is Theorem 9 of [23].

Proposition 3.7. [23] *An odd-cycle inequality* (3.2) *defines a facet of* QP^G *if and only if* $C(V_C,E_C)$ *is a chordless cycle of* G. *When* C *is a chordless cycle of* $K_{m,n}$, *we also have*

$$\mathrm{BQP}^C = \mathrm{BQP}^C_{LP} \cap \left\{\omega \in \mathbb{R}^{2|C|} : \omega \textit{ satisfies all inequalities } (3.2)\right\}.$$

Note that in a complete bipartite graph, $|C| = 4$ if and only if C is chordless. Thus, we obtain the following corollary.

Corollary 3.8. *The inequality* (3.2) *defines a facet for* $\mathrm{BQP}^{m,n}$ *if and only if* $|C| = 4$.

Corollary 3.9. $\mathrm{BQP}^{m,n}$ *has* $2mn(m-1)(n-1)$ *nontrivial facets obtained from* (3.2) *where* $C = C_4$ *and* M *has 1 or 3 elements.*

Proof. $\mathrm{BQP}^{m,n}$ is a Boolean quadric polytope associated with the bipartite graph $K_{m,n}$. For each cycle, there are four choices of M of order 1 and four choices of M of order 3 giving different 8 facets. Since there are $\binom{m}{2}\binom{n}{2} = \frac{m(m-1)n(n-1)}{4}$ C_4's in $K_{m,n}$, $\mathrm{BQP}^{m,n}$ has $2mn(m-1)(n-1)$ nontrivial facets defined by odd-cycle inequalities. $\square$

Proposition 3.10. $\mathrm{BQP}^{2,2}$ *has exactly* 8 *nontrivial facets. Every nontrivial facet is obtained from* (3.2) *where* $C = C_4$ *and* M *has* 1 *or* 3 *elements. More precisely, we have*

$$\mathrm{BQP}^{2,2} = \mathrm{BQP}^{2,2}_{LP} \cap \left\{\omega \in \mathbb{R}^8 : \omega \textit{ satisfies all inequalities } (3.2)\right\}.$$

Therefore, $\mathrm{BQP}^{2,2}$ *has exactly* 24 *facets.*

Proof. From Proposition 3.9, there are 8 nontrivial facets defined by odd-cycle inequalities. Choose $C = K_{2,2}$. From Proposition 3.7, $\mathrm{BQP}^{2,2} = \mathrm{BQP}^{C} = \mathrm{BQP}^{2,2}_{LP} \cap \left\{\omega \in \mathbb{R}^8 : \omega \text{ satisfies all inequalities } (3.2)\right\}$. It implies that all nontrivial facets of $\mathrm{BQP}^{2,2}$ are these 8 odd-cycle inequalities. There are $4mn = 16$ trivial inequalities. Hence, there are exactly 24 facets in total. $\square$

Among all 8 odd-cycle inequalities, 4 of them with $|M| = 1$ are also canonical extension of 4 I_{2222} Bell inequalities for $\mathrm{BQP}^{2,2}$ to $\mathrm{BQP}^{m,n}$. Since all coefficients of y_j's in an odd-cycle inequality are either 0 or -1, only canonical extensions of Bell inequalities I_{2222} can be odd-cycle inequalities. Therefore, for $\mathrm{BQP}^{m,n}$, the $mn(m-1)(n-1)$ canonical extensions of I_{2222} Bell inequalities are the only facet-defining inequalities that are both canonical extensions of Bell inequalities and odd-cycle inequalities.

3.3. **Families of Valid Inequalities Obtained from Rounding Coefficients.** Since the Boolean quadric polytope is well-studied, we investigate where inequalities for it, or the techniques used to generate them, carry over to give valid inequalities and facets for $\mathrm{BQP}^{m,n}$. In particular, we look at techniques based on rounding the coefficients of some families of facets for QP^{m+n}.

We can see that $\mathrm{BQP}^{m,n}$ is a related to QP^{m+n} in the sense that the feasible points for $\mathrm{QP}^{m+n} \subset \mathbb{R}^{m+n+\binom{m+n}{2}}$ restrict to feasible points of $\mathrm{BQP}^{m,n} \subset \mathbb{R}^{m+n+mn}$ using the natural inclusion. However $\mathrm{BQP}^{m,n}$ does not directly inherit valid inequalities or facets from QP^{m+n}.

Consider a valid inequality $\alpha\tau = \alpha^1 u + \alpha^2 v \le \alpha_0$ for QP^{m+n}. Firstly, let us form a related inequality for $\mathrm{BQP}^{m,n}$ by simply attaching the coefficients of $u_1, \ldots, u_m, u_{m+1}, \ldots, u_{m+n}$ to $x_1, \ldots, x_m, y_1, \ldots, y_n$ and those of $v_{i,j}$ to $z_{i,j}$ where applicable. Since many edges (i,j) in K_{m+n} do not appear in $K_{m,n}$, we do not include terms corresponding to these disappearing edges in the corresponding inequality formulated for $\mathrm{BQP}^{m,n}$. Instead of throwing all of these terms away, we replace the variables with positive coefficients by zero and replace the variables with negative coefficient by one. We show that this strategy gives us valid inequalities, but they are not necessarily facet-defining.

Definition 3.11. Denote $V = \{1, \ldots, m, m+1, \ldots, m+n\}$, $G(V,E) = K_{m,n}$ and $G'(V,E') = K_{m+n}$.

Let (u,v) be a vertex of QP^{m+n}, a polytope corresponding to G', where $u \in \{0,1\}^{m+n}$ and $v \in \{0,1\}^{(m+n)(m+n-1)/2}$. Then the *bipartite restriction* of (u,v) is $(\tilde{x}, \tilde{y}, \tilde{z})$ where $\tilde{x}$ is obtained from the first m entries of u, $\tilde{y}$ is obtained from the last n entries of u and $\tilde{z}$ is obtained from v by discarding entries corresponding to $(i,j) \in E' \backslash E$.

Let $\alpha\tau = \alpha^1 u + \alpha^2 v \le \alpha_0$ be a valid inequality for QP^{m+n}. Denote $E^- = \{(i,j) \in E' \backslash E : \alpha^2_{ij} < 0\}$. The *rounded inequality* of $\alpha\tau \le \alpha_0$ is

$$\tilde{a}\tilde{\omega} = \tilde{a}^1\tilde{x} + \tilde{a}^2\tilde{y} + \tilde{a}^3\tilde{z} \le \alpha_0 - \sum_{ij \in E^-} \alpha^2_{ij}, \tag{3.3}$$

where $\tilde{a}^1$ is obtained from the first m entries of α^1, $\tilde{a}^2$ is obtained from the last n entries of α^1 and $\tilde{a}^3$ is obtained from α^2 by discarding entries $(i,j) \in E' \backslash E$.

The following proposition is easy to verify.

Proposition 3.12. *The inequality* (3.3) *is valid for* $\mathrm{BQP}^{m,n}$.

Clique inequalities and cut inequalities are the main families of non-trivial facet-defining inequalities for the Boolean quadric polytope found by Padberg [23]. We remark that while an arbitrary non-trivial clique or cut in K_n yields a facet-defining inequality for QP^{m+n}, inequalities obtained by discarding the terms corresponding to the missing edges in complete bipartite graph are not even valid. Moreover, we do not get any facet-defining inequalities for $\mathrm{BQP}^{m,n}$ from rounding applied to these inequalities. Also, we give an alternative modification on a family of rounded inequalities, but they are not facet-defining. Some of these inequalities are tight, but some of them are not tight anywhere.

We introduce some notation for describing inequalities for QP^{m+n} and $\mathrm{BQP}^{m,n}$. Let $G(V,E)$ be an underlying graph of the polytope we consider, i.e. K_{m+n} or $K_{m,n}$ respectively. Let S and T be disjoint subsets of V. For any variable $\zeta_{i,j}$ whose index corresponds to an edge in E, denote $\zeta(S:T) = \sum_{i\in S, j\in T}\zeta_{i,j}$. In case that $S=\{s\}$, we can use $\zeta(s:T)$ instead of $\zeta(\{s\}:T)$ for convenience. The same applies for the case when T is a singleton. Together with notations given in Section 3.2, we define various families of valid inequalities.

3.3.1. *Rounded Clique Inequalities.* Here we obtain a family of valid inequalities for $\mathrm{BQP}^{m,n}$ from rounding the family of clique inequalities for QP^{m+n}. Let $G'(V,E') = K_{m+n}$. Let $S \subseteq V$ with $|S| \geq 3$ and α be an integer in $\{1,\dots,|S|-2\}$. The original clique inequality for QP^{m+n} given by Padberg [23] is in the form

$$Cq(\tau) := \alpha u(S) - v(E(S)) \leq \frac{\alpha(\alpha+1)}{2}.$$

This inequality defines a facet of QP^{m+n}.

We consider $G(V,E) = K_{m,n}$. Let $S_1 = S\cap I$ and $S_2 = S \cap J$. Let $S\subseteq V$ where $|S|\geq 3$ and α be an integer such that $1 \leq \alpha \leq |S|-2$. The rounded clique inequality is

$$RCq(\omega) := \alpha x(S_1) + \alpha y(S_2) - z(S_1:S_2) \leq \beta, \tag{3.4}$$

where $\beta = \alpha(\alpha+1)/2 + |S_1|(|S_1|-1)/2 + |S_2|(|S_2|-1)/2$. These $\displaystyle\sum_{s=3}^{m+n}(s-2)\binom{m+n}{s}$ inequalities are valid for $\mathrm{BQP}^{m,n}$, but they are not facet-defining.

Theorem 3.13. *The inequality* (3.4) *does not define a facet for* $\mathrm{BQP}^{m,n}$ *and is valid for* $\mathrm{BQP}^{m,n}_{LP}$. *Moreover,* (3.4) *is tight only when* S_1 *or* S_2 *is a singleton and* $\alpha = |S|-2$.

Proof. Let $s := |S|$, $h := |S_1|$ and $k := |S_2|$. Thus, $s = h+k$, and we assume without loss of generality that $h \leq k$. The proof is split into three cases depending on the values of h and k.

Case 1: $h=0$ and $k=s$. Here $S_1=\emptyset$ and the terms $z(S_1:S_2)$ disappear. By Proposition 2.8, (3.4) is not facet defining. Then the inequality reduces to

$$(k-2)y(S_2) \leq (k-1)^2. \tag{c1}$$

Consider the sum of inequalities $(k-2)y_j \leq k-2$ over all $j \in S_2$. We obtain $(k-2)y(S_2) \leq (k-2)k$. Since $(k-1)^2-(k-2)k > 0$, (c1) is also valid for $\mathrm{BQP}^{m,n}_{LP}$.

Let $R \subseteq V$ be the set of vertices corresponding to a feasible solution $y \in \mathrm{BQP}^{m,n}$. The left-hand side reaches its maximum value $(k-2)k$ when $R\cap S = S_2$. Since $(k-1)^2-(k-2)k>0$, there are no vertices ω^R of $\mathrm{BQP}^{m,n}$ where $RCq(\omega^R)=\beta$, which means (3.4) is not tight when S_1 or S_2 is empty.

Case 2: $h=1$ and $k=s-1$. Here S_1 is a singleton, say $S_1=\{u\}$. We construct a non-trivial linear combination of the constraints of $\mathrm{BQP}^{m,n}$ whose left-hand side matches that of (3.4). Consider the sum of inequalities $x_u+y_j-z_{u,j}\leq 1$ over all $j\in S_2$, $(\alpha-k)x_u \leq 0$ and $(\alpha-1)y_j \leq \alpha-1$ for all $j \in S_2$. Then we get

$$\alpha x(S_1) + \alpha y(S_2) - z(S_1:S_2) \leq k + k(\alpha-1) = \alpha k. \tag{c2}$$

By construction, (c2) is valid for $\mathrm{BQP}^{m,n}$ and $\mathrm{BQP}^{m,n}_{LP}$. Moreover, the expression $\beta - \alpha k$ is nonnegative since $2(\beta - \alpha k) = (\alpha - k + 1)(\alpha - k) \geq 0$, which means its right-hand side is no larger than that of (3.4). It follows that (3.4) is valid for $\mathrm{BQP}^{m,n}_{LP}$ and redundant for $\mathrm{BQP}^{m,n}$. Therefore, it is not facet-defining for $\mathrm{BQP}^{m,n}$. Furthermore, when $R \cap S = S_2$ and $\alpha = k - 1 = s - 2$, (c2) holds with equality and $\beta = \alpha k$. Therefore, (3.4) is tight in this case.

Case 3: $h, k \geq 2$**.** Here we consider three different subcases. For the first two subcases, we follow the same approach as in Case 2. We construct a valid inequality for $\mathrm{BQP}^{m,n}$, whose left-hand side is the same as that of (3.4), from a linear combination of the constraints of $\mathrm{BQP}^{m,n}$. We later show that the right-hand side of the constructed inequality is at most that of (3.4). It leads to the fact that (3.4) is redundant for $\mathrm{BQP}^{m,n}$, and hence, does not define a facet. Furthermore, it also implies that (3.4) is valid for $\mathrm{BQP}^{m,n}_{LP}$.

Case 3.1: $h \leq k < \alpha$**.** We sum over all hk inequalities in the form $x_i + y_j - z_{i,j} \leq 1$ where $(i,j) \in (S_1 : S_2)$. Since each x_i appears k times and each y_j appears h times in this sum, we add the inequalities $(\alpha - k)x_i \leq \alpha - k$ for all $i \in S_1$ and $(\alpha - h)y_j \leq \alpha - h$ for all $j \in S_2$. Then we get

$$\alpha x(S_1) + \alpha y(S_2) - z(S_1 : S_2) \leq hk + h(\alpha - k) + k(\alpha - h) = \alpha h + \alpha k - hk.$$

Its right-hand side $\alpha h + \alpha k - hk$ is at most that of (3.4) since

$$2(\beta - \alpha h - \alpha k + hk) = (\alpha - h - k + 1)(\alpha - h - k) \geq 0. \tag{g1}$$

Therefore, (3.4) is valid for $\mathrm{BQP}^{m,n}_{LP}$, but it is not a facet-defining inequality for $\mathrm{BQP}^{m,n}$. However, we can see from (g1) that $\beta = \alpha h + \alpha k - hk$ if and only if $\alpha = h + k = s$ or $\alpha = h + k - 1 = s - 1$. Since $\alpha \leq s - 2$, there is no ω^R where $RCq(\omega^R) = \beta$ in Case 1.

Case 3.2: $h < \alpha \leq k$**.** We again consider the sum of $x_i + y_j - z_{i,j} \leq 1$ where $(i,j) \in (S_1 : S_2)$ and add the inequalities $(\alpha - k)x_i \leq 0$ for all $i \in S_1$ and $(\alpha - h)y_j \leq \alpha - h$ for all $j \in S_2$ to obtain

$$\alpha x(S_1) + \alpha y(S_2) - z(S_1 : S_2) \leq hk + k(\alpha - h) = k\alpha,$$

which is a valid inequality for $\mathrm{BQP}^{m,n}$ whose right-hand side αk does not exceed that of (3.4) since

$$2(\beta - \alpha k) = (\alpha - k)(\alpha - k - 1) + h(h - 1) \geq 0. \tag{g2}$$

Then inequality (3.4) does not define a facet for $\mathrm{BQP}^{m,n}$, but it is still valid for $\mathrm{BQP}^{m,n}_{LP}$. From (g2), we can see that if $\beta = \alpha k$, then $h(h-1)$ must be zero. Since $h \geq 2$, (3.4) is also not tight in this case.

Case 3.3: $\alpha \leq h \leq k$**.** Let

$$U := \begin{cases} \frac{-1+\sqrt{1+4(s-(h-k)^2)}}{2} & ;\ \text{if } s - (h-k)^2 \geq 0 \\ 0 & ;\ \text{otherwise} \end{cases} \quad \text{and } L := \frac{2s - 1 - \sqrt{8h(k-1)+1}}{2}.$$

For this case, we show the three following claims.

Claim 1 For a given $s = h + k > 0$, there exists no $\alpha \in \mathbb{Z}$ such that $L < \alpha < U$.

Claim 2 (3.4) is not facet-defining if $\alpha \geq U$.

Claim 3 (3.4) is not facet-defining if $\alpha \leq L$.

Claim 1 implies that for a given s, $\alpha \leq L$ or $\alpha \geq U$. Then from **Claim 2** and **Claim 3**, (3.4) with this α does not define a facet, allowing us to conclude Theorem 3.13. It remains to prove these claims.

Proof of **Claim 1**. Since we assume that $h, k \geq 2$, S has size at least four. It is trivial to verify that **Claim 1** holds for $4 \leq s \leq 6$. Thus, we can assume that $s \geq 7$. We first consider the polynomial $p(s) := s^4 - 8s^3 + 10s^2 - 8s + 1$ which is increasing and non-negative for $s \geq 7$. Then we obtain

$$\begin{aligned} 0 &\leq s^4 - 8s^3 + 10s^2 - 8s + 1 = (s^2 + 4s - 1)^2 - 4s^2(4s+1) \\ 2s\sqrt{4s+1} &\leq s^2 + 4s - 1 \\ 2s^2 - 4s + 3 &\leq 4s^2 - 4s\sqrt{4s+1} + 4s + 1 = (2s - \sqrt{4s+1})^2 \\ \sqrt{4s+1} &\leq 2s - \sqrt{2s^2 - 4s + 3} \leq 2s - \sqrt{8h(k-1)+1}. \end{aligned}$$

It follows that

$$U \leq \frac{\sqrt{4s+1}-1}{2} \leq \frac{2s-1-\sqrt{8h(k-1)+1}}{2} = L.$$

Thus, we get the claim.

To show **Claim 2** and **Claim 3**, we construct valid inequalities whose left-hand side of these inequalities and that of (3.4) are the same, but the right-hand side of the constructed inequalities are not greater than that of (3.4). As for **Claim 2**, we use the sum of $x_i + y_j - z_{i,j} \leq 1$ where $(i,j) \in (S_1 : S_2)$, $(\alpha - k)x_i \leq 0$ for all $i \in S_1$ and $(\alpha - h)y_j \leq 0$ for all $j \in S_2$. This yields

$$\alpha x(S_1) + \alpha y(S_2) - z(S_1 : S_2) \leq hk. \tag{c3}$$

Since $\alpha \geq U$, $2(\beta - hk) = \alpha^2 + \alpha + (h-k)^2 - s \geq 0$. Thus, (3.4) does not define a facet for $\mathrm{BQP}^{m,n}$, but it is a valid inequality for $\mathrm{BQP}^{m,n}_{LP}$.

We consider a different sum of valid inequalities to prove **Claim 3**. Let $S_1 = \{1, \ldots, h\}$ and $S_2 = \{1, \ldots, k\}$. Take the sum of $x_i + y_i - z_{i,i} \leq 1$ for $i = 1, \ldots, h$, $(\alpha - 1)x_i \leq \alpha - 1$ for all $i = 1, \ldots, h$, $(\alpha - 1)y_j \leq \alpha - 1$ for all $j = 1, \ldots, h$, $\alpha y_j \leq \alpha$ for $j = h+1, \ldots, k$, and $-z_{i,j} \leq 0$ for all $i = 1, \ldots, h$ and $j = 1, \ldots, k$ where $i \neq j$. The sum of these inequalities becomes

$$\alpha x(S_1) + \alpha y(S_2) - z(S_1 : S_2) \leq h + h(\alpha - 1) + h(\alpha - 1) + \alpha(k-h) = \alpha s - h. \tag{c4}$$

Since $\alpha \leq L$, we have $2(\beta - \alpha s + h) = \alpha^2 + (1-2s)\alpha + k(k-1) + (h+1)h \geq 0$. It leads to the fact that (3.4) does not facet-defining for $\mathrm{BQP}^{m,n}$, while it is valid for $\mathrm{BQP}^{m,n}_{LP}$. Thus, we get the claim.

Suppose that (3.4) is tight, which means (c3) and (c4) are also tight. **Claim 1** states that $\alpha \geq U$ or $\alpha \leq L$, so we can consider it in two cases.

Case 3.3.1: $\alpha \geq U$. If (c3) is tight, there exists ω^R satisfying (c3) with equality and there exist S_1 and S_2 such that $\beta = hk$. It follows that $x_i^R + y_j^R - z_{i,j}^R = 1$ for all $(i,j) \in (S_1 : S_2)$, $(\alpha - k)x_i^R = 0$ for all $i \in S_1$ and $(\alpha - h)y_j^R = 0$ for all $j \in S_2$.

When $\alpha \neq h$ and $\alpha \neq k$, we have $x_i^R = 0$ for all $i \in S_1$ and $y_j^R = 0$ for all $j \in S_2$. It follows that $z_{i,j}^R = x_i^R y_j^R = 0$ for all $(i,j) \in (S_1 : S_2)$. Hence, $x_i^R + y_j^R - z_{i,j}^R = 0 < 1$ for all $(i,j) \in (S_1 : S_2)$, a contradiction.

If $\alpha = h$ or $\alpha = k$, since $\alpha \leq h \leq k$, in both cases, we obtain $\alpha = h$. Since $\alpha^2 + \alpha + (h-k)^2 - s = 2(\beta - hk) = 0$ and $\alpha \geq U$, we get $\alpha = U$. From $h = \alpha = U$, we get that $h = (2k + \sqrt{8k - 4k^2})/4$. Since h is a real number, we have $0 \leq k \leq 2$. Since we assume that $k \geq 2$, it implies that $k = 2$ and it yields $h = 1$ which contradicts the assumption that $h \geq 2$.

Case 3.3.2: $\alpha \leq L$. When (c4) is tight, there exists ω^R satisfying (c4) with equality and there exist S_1 and S_2 such that $\beta = \alpha s - h$. Thus, $x_i^R + y_i^R - z_{i,i}^R = 1$ for $i = 1, \ldots, h$, $(\alpha - 1)x_i^R = \alpha - 1$ for all $i = 1, \ldots, h$, $(\alpha - 1)y_j^R = \alpha - 1$ for all $j = 1, \ldots, h$, $\alpha y_j^R = \alpha$ for $j = h+1, \ldots, k$, and $-z_{i,j}^R = 0$ for all $i = 1, \ldots, h$ and $j = 1, \ldots, k$ where $i \neq j$.

If $\alpha \neq 1$, we have $x_i^R = 1$ for all $i = 1, \ldots, h$ and $y_j^R = 1$ for all $j = 1, \ldots, k$. Since $h, k \geq 2$, there exists $z_{i,j}$ where $(i,j) \in (S_1 : S_2)$ and $i \neq j$ such that $-z_{i,j} = -x_i y_j = -1 < 0$, a contradiction.

When $\alpha = 1$, note that $\alpha^2 + (1-2s)\alpha + k(k-1) + (h+1)h = 2(\beta - \alpha s + h) = 0$ and $\alpha \leq L$ implies that $\alpha = L$. From $L = \alpha = 1$, we get that $h(h-1) + (k-1)(k-2) = 0$. Since $h, k \geq 2$, we have $h(h-1) > 0$ and $(k-1)(k-2) \geq 0$, a contradiction.

Therefore, we conclude that there are no ω^R such that $RCq(\omega^R)$ holds with equality for Case 3.3 as well. □

3.3.2. *Rounded Cut Inequalities.* Different from the clique inequalities, the cut inequalities involve two disjoint sets of vertices. Let $G'(V, E') = K_{m+n}$, and let S and T be disjoint subsets of V with $|S| \geq 1$ and $|T| \geq 2$. The cut inequality for QP^{m+n} given by Padberg [23] is

$$Cut(\tau) := -u(S) - v(E(S)) + v(S : T) - v(E(T)) \leq 0,$$

defining a facet for QP^{m+n}. If we consider instead the bipartite graph $G(V,E)=K_{m,n}$ and obtain a new valid inequality for $\mathrm{BQP}^{m,n}$ by dropping the terms corresponding to edges in $E'\backslash E$, the inequality will no longer be valid.

Let S and T be disjoint subsets of V with $|S|\geq 1$ and $|T|\geq 2$. Denote $S_1=S\cap I$, $S_2=S\cap J$, $T_1=T\cap I$ and $T_2=T\cap J$. The rounded version of the cut inequality is

$$RCut(\omega) := -x(S_1)-y(S_2)-z(S_1:S_2)+z(S_1:T_2)+z(T_1:S_2)-z(T_1:T_2)\leq\beta, \tag{3.5}$$

where $\beta=\binom{|S_1|}{2}+\binom{|S_2|}{2}+\binom{|T_1|}{2}+\binom{|T_2|}{2}$. This rounded inequality is valid for $\mathrm{BQP}^{m,n}$. There are $\displaystyle\sum_{s=1}^{m+n-2}\sum_{t=2}^{m+n-s}\binom{m+n}{s}\binom{m+n-s}{t}$ inequalities in this form.

We can strengthen this inequality by changing the constant on the right hand side. The strengthened version is

$$SRCut(\omega) := -x(S_1)-y(S_2)-z(S_1:S_2)+z(S_1:T_2)+z(T_1:S_2)-z(T_1:T_2)\leq\gamma, \tag{3.6}$$

where $\gamma=|S_1|(|T_2|-1)+|S_2|(|T_1|-1)$. This inequality is valid for $\mathrm{BQP}^{m,n}$ but it does not define a facet. Note that for $a,b\in\mathbb{Z}^+$. $a(b-1)\leq\binom{a}{2}+\binom{b}{2}$. Hence, inequality (3.6) is stronger than (3.5).

Theorem 3.14. *The inequality* (3.6) *is valid for $BQP^{m,n}$ but is not facet-defining. Moreover, there are no vertices $\omega\in\mathrm{BQP}^{m,n}$ where $SRCut(\omega)=\gamma$.*

Proof. Let $u\in T_1$ and $v\in T_2$. Consider the left hand side of (3.5) which is

$$\begin{aligned}&-x(S_1)-y(S_2)-z(S_1:S_2)+z(S_1:T_2)+z(T_1:S_2)-z(T_1:T_2)\\&\quad=-x(S_1)-z(S_1:S_2)+z(S_1:v)+z(S_1:T_2\backslash\{v\})-y(S_2)-z(T_1:T_2)+z(u:S_2)+z(T_1\backslash\{u\}:S_2).\end{aligned}$$

Adding

$$\begin{aligned}-x_i+z_{i,v}&\leq 0, && i\in S_1,\\-y_j+z_{u,j}&\leq 0, && j\in S_2,\\z_{i,j}&\leq 1, && i\in S_1, j\in T_2\backslash\{v\} \text{ or } i\in T_1\backslash\{u\}, j\in S_2,\\-z_{i,j}&\leq 0 && i\in S_1, j\in S_2 \text{ or } i\in T_1, j\in T_2\end{aligned}$$

together, we obtain

$$\begin{aligned}&-x(S_1)-z(S_1:S_2)+z(S_1:v)+z(S_1:T_2\backslash\{v\})\\&-y(S_2)-z(T_1:T_2)+z(u:S_2)+z(T_1\backslash\{u\}:S_2)\leq|S_1|(|T_2|-1)+|S_2|(|T_1|-1).\end{aligned}$$

Therefore, (3.6) is valid for $\mathrm{BQP}^{m,n}$ and $\mathrm{BQP}^{m,n}_{LP}$ but does not define a facet.

Let $\omega=(x,y,z)$ be a vertex in $\mathrm{BQP}^{m,n}$. Consider an edge (p,q) where $p\in S_1$ and $q\in S_2$. If $z_{p,q}=1$, then $-z_{p,q}=-1<0$. Thus, $SRCut(\omega)<\gamma$. If $z_{p,q}=0$, then x_p or y_q must be zero. Without loss of generality, we assume that $x_p=0$. Then $z_{p,j}=0<1$ for all $j\in T_2\backslash\{v\}$. So we obtain $SRCut(\omega)<\gamma$ as well. □

4. Conclusion

In this study, we investigated the bipartite Boolean quadratic programming problem from a polyhedral point of view. We investigated the polytope $\mathrm{BQP}^{m,n}$ arising from a linearization of an integer programming formulation of BQP01. When the integrality constraints are relaxed for some variables, we can show that the polytopes $\mathrm{BQP}^{m,n}_x$, $\mathrm{BQP}^{m,n}_y$ and $\mathrm{BQP}^{m,n}_z$ obtained from these partial relaxations are the same as $\mathrm{BQP}^{m,n}$.

Various strategies to establish new classes of valid inequalities and facet-defining inequalities are presented. In particular, we obtain families of trivial facets and odd-cycle inequalities directly from the corresponding families for QP^{m+n}. Rounding techniques are applied to families of clique inequalities and cut inequalities for QP^{m+n}. In summary, here are the list of valid inequalities and facet-defining inequalities established in our work.

- **Trivial facets (1.6)-(1.9):** $4mn$ facet-defining inequalities for both $\mathrm{BQP}^{m,n}$ and $\mathrm{BQP}^{m,n}_{LP}$.
- **Canonical extensions of Bell inequalities (2.1):** $mn(m-1)(n-1)+\sum_{l=3}^{m}(n\binom{m}{l}+m\binom{n}{l})l!$ non-trivial facet-defining inequalities, $mn(m-1)(n-1)$ of which are also odd-cycle inequalities.
- **Odd-cycle inequalities (3.2):** $\displaystyle\sum_{i=2}^{m}\binom{m}{i}\binom{n}{i}(\sum_{j=1}^{i}\binom{2i}{2j-1})$ valid inequalities, of which $2mn(m-1)(n-1)$ are facet-defining and $mn(m-1)(n-1)$ of them are also canonical extensions of I_{2222} Bell inequalities.
- **Rounded clique inequalities (3.4):** $\displaystyle\sum_{s=3}^{m+n}(s-2)\binom{m+n}{s}$ valid inequalities for both $\mathrm{BQP}^{m,n}$ and $\mathrm{BQP}^{m,n}_{LP}$, none of which are facet-defining.
- **Rounded cut inequalities (3.5):** $\displaystyle\sum_{s=1}^{m+n-2}\sum_{t=2}^{m+n-s}\binom{m+n}{s}\binom{m+n-s}{t}$ valid inequalities for both $\mathrm{BQP}^{m,n}$ and $\mathrm{BQP}^{m,n}_{LP}$, none of which are facet-defining.
- **Strengthened rounded cut inequalities (3.6):** $\displaystyle\sum_{s=1}^{m+n-2}\sum_{t=2}^{m+n-s}\binom{m+n}{s}\binom{m+n-s}{t}$ valid inequalities for both $\mathrm{BQP}^{m,n}$ and $\mathrm{BQP}^{m,n}_{LP}$, none of which are facet-defining.

We conclude that the extension of Bell inequalities and some odd cycle inequalities define facets and are potentially useful for branch-and-cut algorithms. On the other hand, analogues of the rounded cut and clique inequalities, which are important sources of cuts for QP^n, turn out to be valid for $\mathrm{BQP}^{m,n}_{LP}$, and, thus, are not helpful here.

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