A CLAWPACK Implementation for a Model of 2-Class Traffic Flow

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1. A Single-Class Traffic Model

Macroscopic 1-D traffic model assumptions:

- Quantify traffic by car density $\rightarrow \rho(x,t)$.
- Conserve number of cars $\rightarrow$ Conservation Law (PDE).
- Velocity, $v(\rho)$, modeled as a function of density only.
- Flux, $f(\rho) = v(\rho)\rho$, cars passing cross-section of road / unit time.

1-Class Equations

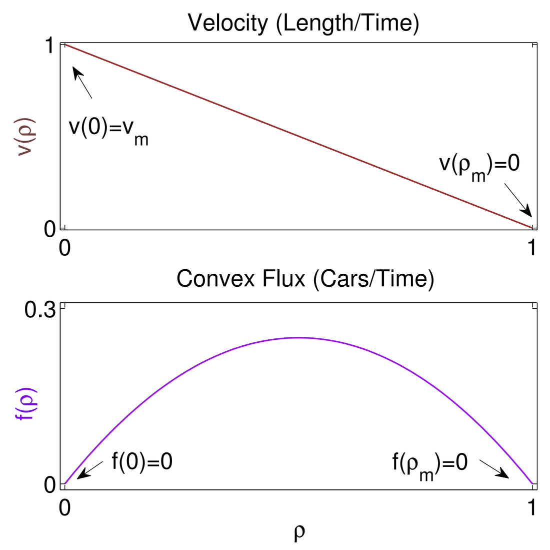
$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x} f(\rho(x,t)) = 0$$

$$f(\rho) = v(\rho)\rho = v_m\left(1 - \frac{\rho}{\rho_m}\right)\rho$$

$\rho(x,t) =$ Density (cars/length)

$v_m =$ Maximum velocity $= 1$

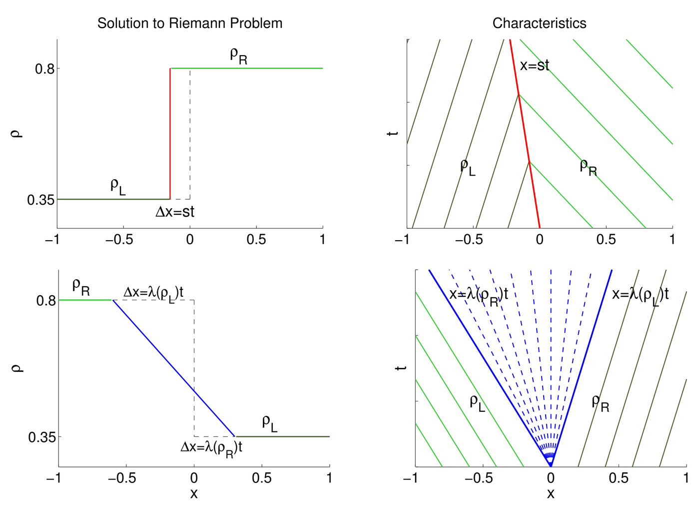
$\rho_m =$ Maximum density $= 1$



1.1 Riemann Problem: hyperbolic conservation law with piecewise constant initial condition

$$\rho(x,0) = \begin{cases} \rho_L & : x \le 0 \\ \rho_R & : x > 0 \end{cases}$$

- Characteristic speed $\lambda(\rho) = f'(\rho) = v_m\left(1 - 2\frac{\rho}{\rho_m}\right)$
- With convex flux, solution has 1 elementary wave of two types.



Shock, cars in back move faster

- Speed s from Rankine-Hugoniot condition (RHC) $\rightarrow s(\rho_L - \rho_R) = f(\rho_L) - f(\rho_R)$
- Entropy condition $\rightarrow \lambda(\rho_L) > s > \lambda(\rho_R)$

Rarefaction, cars in back move slower

- $\lambda(\rho_L) < \lambda(\rho_R)$.
- Fan of rays in rarefaction zone.

2. A Two-Class Traffic Model

- Different vehicle classes obey different velocity functions.
- ρ_1 is the density of fast drivers and ρ_2 is the density of slow drivers.

2-Class Equations

$$\frac{\partial}{\partial t}\vec{q}(x,t) + \frac{\partial}{\partial x}\vec{f}(\vec{q}(x,t)) = 0$$

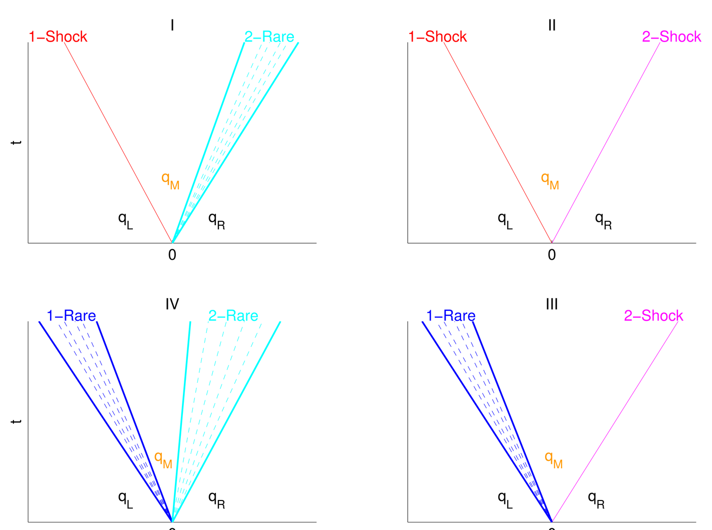
$$\vec{q} = \begin{bmatrix}\rho_1 \\ \rho_2\end{bmatrix}, \quad \vec{f}(\vec{q}) = \begin{bmatrix}(1-\rho)\rho_1 \\ \alpha(1-\rho)\rho_2\end{bmatrix}$$

$\alpha =$ ratio of max velocities ≤ 1

$\rho = \rho_1 + \rho_2$, total density

$\lambda_1 < \lambda_2$, eigenvalues of Jacobian of $\vec{f}$

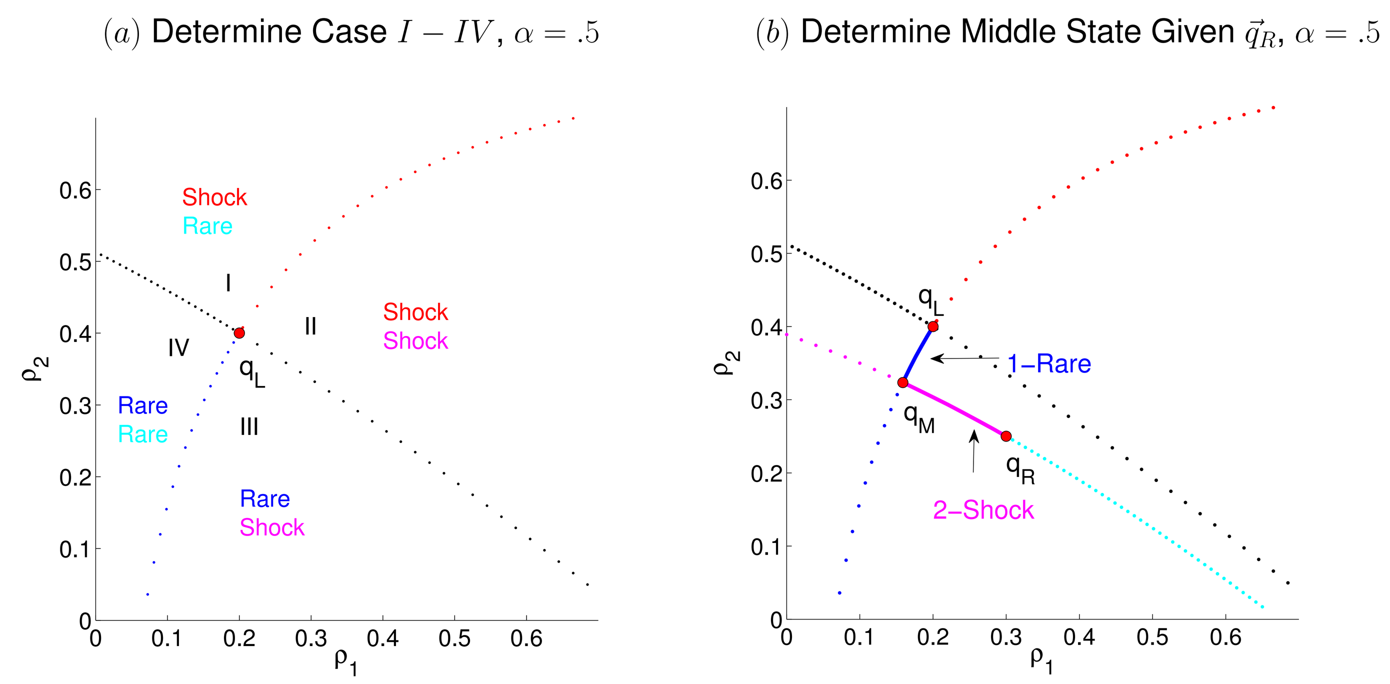
Cases $(I - IV)$



2.1 Riemann Problem

- 2 elementary waves separated by a constant middle state $\vec{q}_M \rightarrow$ 4 cases.
- Left-most wave determined by λ_1; Right-most wave determined by λ_2.
- $\vec{q}_M$ for shocks satisfy a vector RHC.
- $\vec{q}_M$ for rarefactions satisfy a Riemann invariant condition.

- 1-curve: Possible middle states $\vec{q}_M$ that connect to $\vec{q}_L$ by a 1-rarefaction/shock.
- 2-curve: Possible middle states $\vec{q}_M$ that connect to $\vec{q}_R$ by a 2-rarefaction/shock.



- (a) 1-curve and 2-curve create 4 regions, where position $\vec{q}_R$ determines case $I - IV$.
- (b) Constant middle state $\vec{q}_M$ is the intersection of the 1-curve and 2-curve.

3. Finite Volume Method

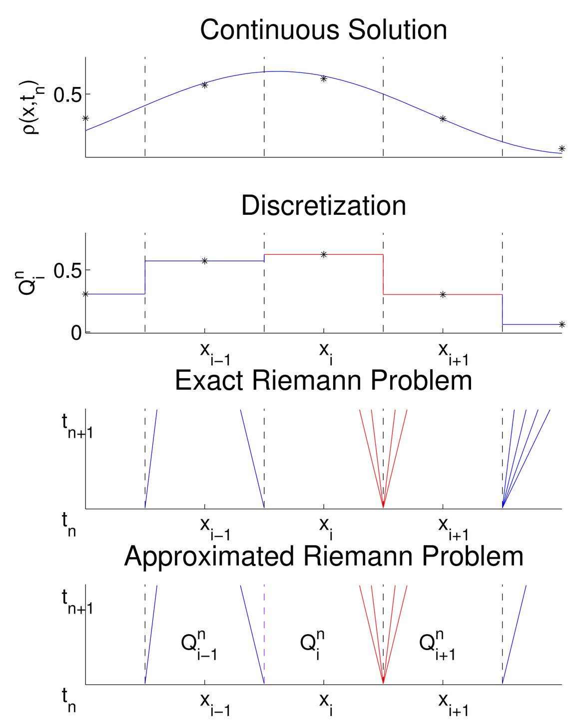
- $Q_i^n \rightarrow$ Average density in a cell:
- $F_{i-1/2}^n \rightarrow$ Average flux across cell edge:

$$F_{i-1/2}^n = f(Q_{i-1/2})$$

- Update $\rightarrow Q_i^{n+1} = Q_i^n + \underbrace{\frac{\Delta t}{\Delta x}(F_{i-1/2}^n - F_{i+1/2}^n)}_{\text{Net change in density in cell } x_i}$

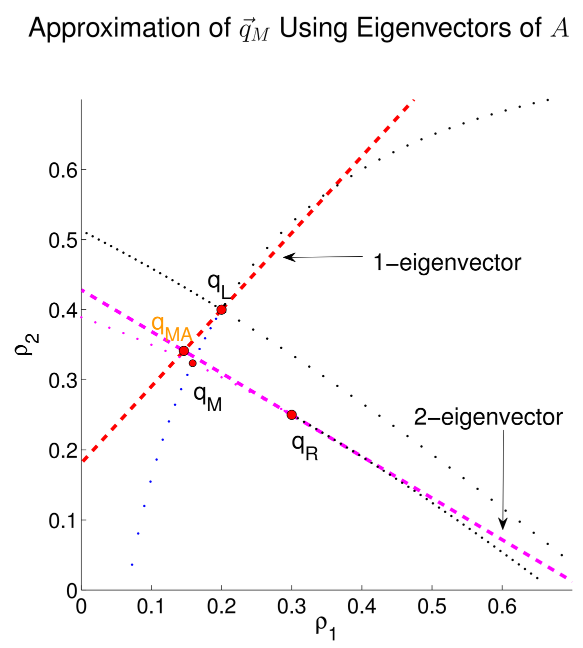
- $Q_{i-1/2}$ from an approximate Riemann problem.

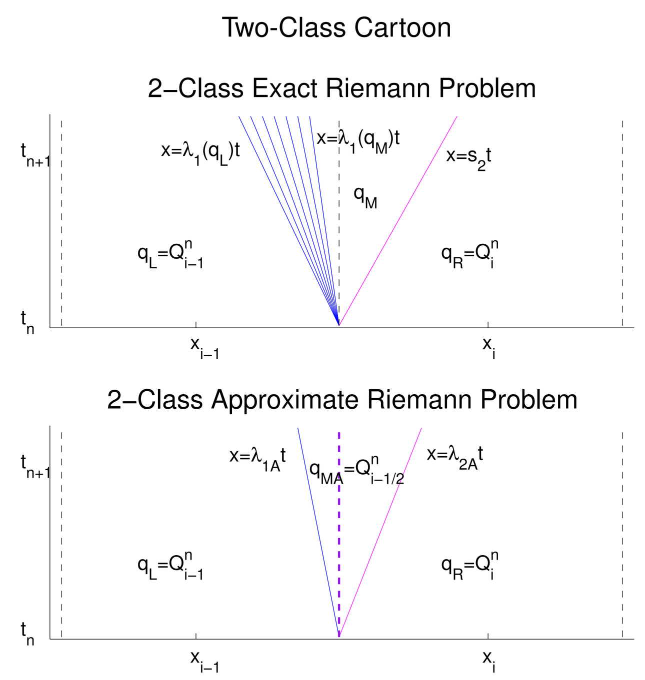
Single-Class Cartoon



3.1 Roe's Approximate Riemann Solver

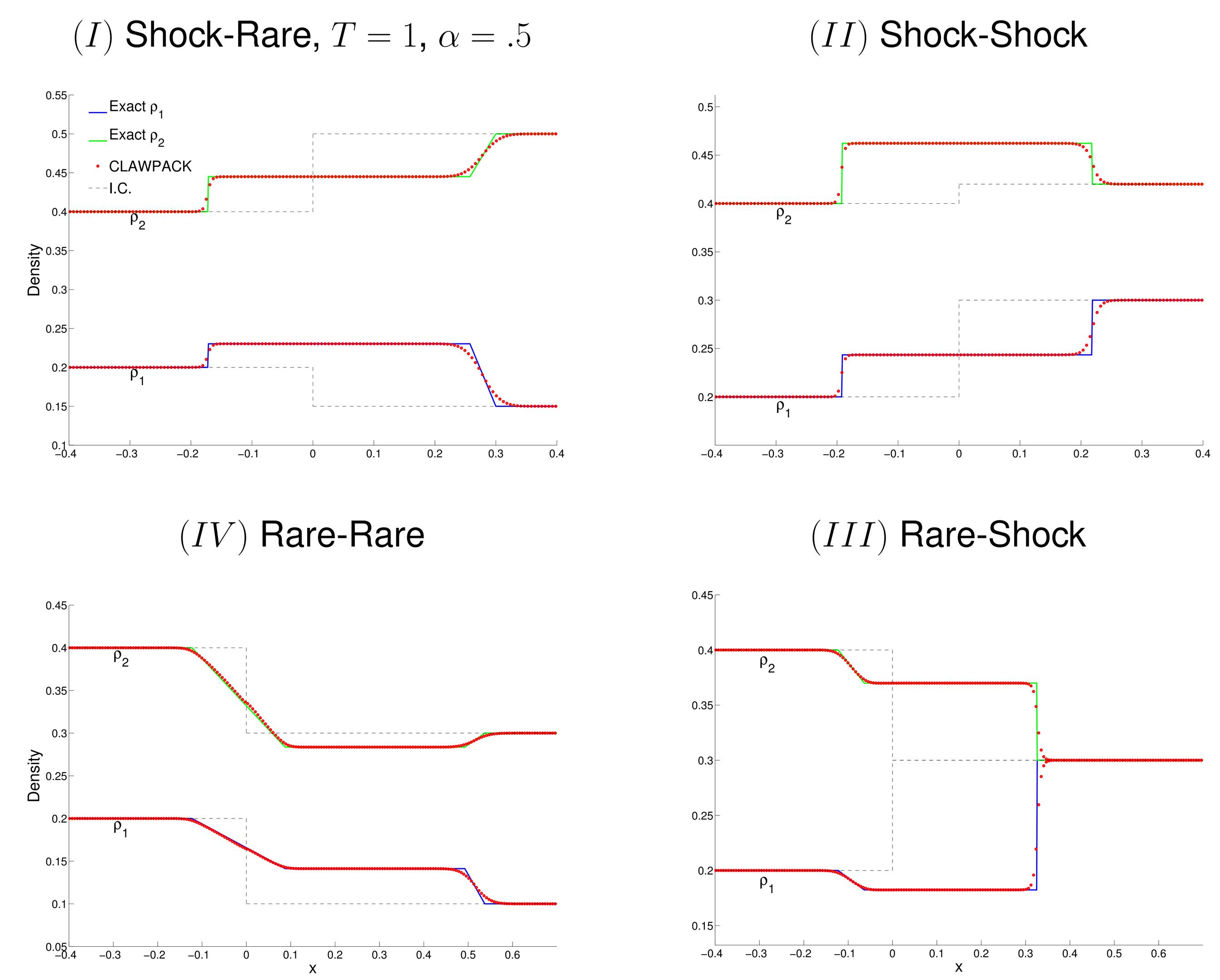
- Finding exact $f(Q_{i-1/2})$ can be expensive $2(b)$.
- Approximate the 1 and 2 curves in $2(b)$ by sensible straight lines.
- Roe Linearization $\rightarrow$ Lines from eigenvectors of an approximate Jacobian matrix A
- Eigenvalues of A should be consistent with shock speeds.
- $A =$ mean value integral of $\nabla_{\vec{q}}\vec{f}$ along path $\vec{q}_L - \vec{q}_R$.



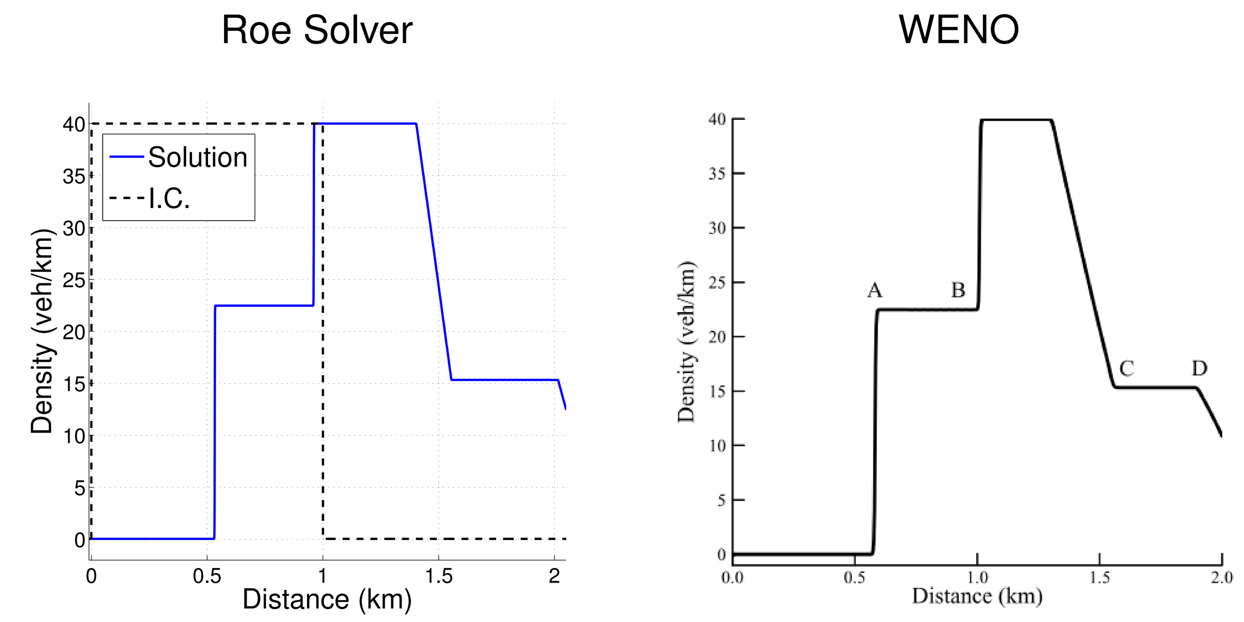


4. CLAWPACK Implementation

- CLAWPACK: package using Riemann solvers to compute hyperbolic equations.
- Exact and numerical solutions to Riemann problem for cases $(I - IV)$.



- Test Roe Solver against benchmark solution from a WENO scheme (2).



5. Future work

(1) Extend to N-class traffic flow.

(2) Study different velocity functions.

6. References

1. Randall J. LeVeque. *Finite Volume Methods for Hyperbolic Problems.* Cambridge University Press, 2002.
2. Mengping Wang, Chi-Wang Shu, George C. K. Wong, and S. C. Wong.*Journal of computational Physics*, 191:639-659, 2003.