Discrete Spectrum of the Rotating Shallow Water Eigenproblem

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Overview

- Singular eigenvalue problems: discrete/continuous spectrum
- Atmospheric fluid dynamics: rotating shallow water (RSW)
- Equatorial wave propagation: large-scales penetrate, small-scales absorbed
- My contribution (Numerics, WKB)
  - Infinite number of discrete modes
  - How "large" for equatorial penetration

Singular eigenvalue problems

- Standard eigenvalue problems have a discrete spectrum (countably infinite modes)
- Singular points can also generate a continuous spectrum (uncountably infinite modes)
- Example: 2d Euler (Couette flow) [2, 4]

$$\left(y\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)\nabla^2\Psi(x,y,t)=0$$

$$\Psi(x,0,t)=\Psi(x,1,t)=0$$

  - No discrete spectrum solutions
  - Continuous spectrum solutions

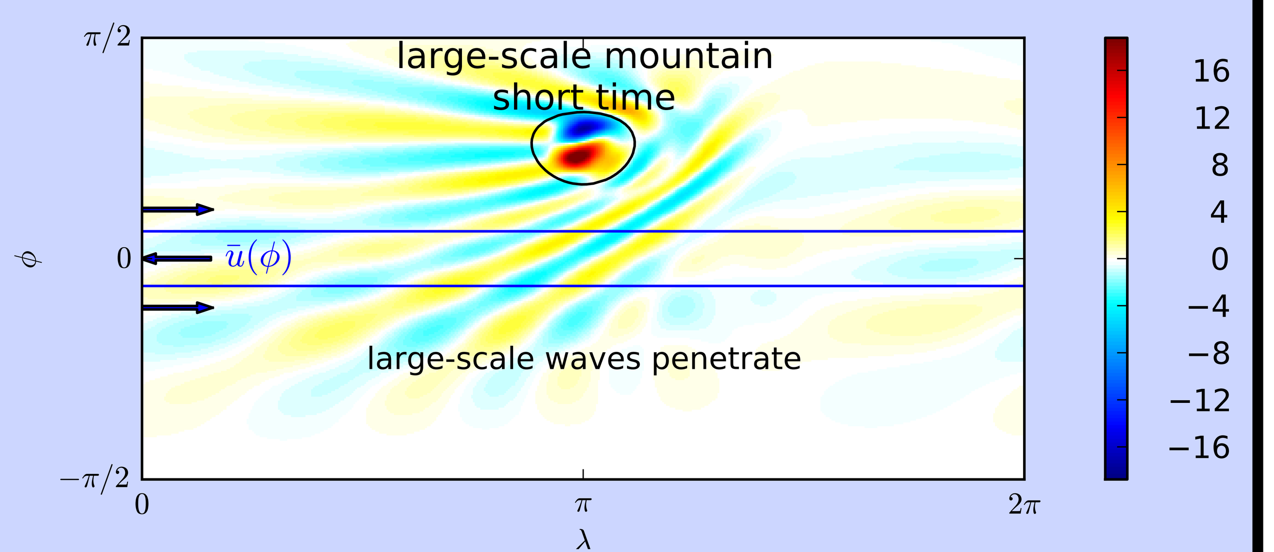
$$im(y-c)\nabla^2\hat{\Psi}(y)e^{im(x-ct)}=0$$

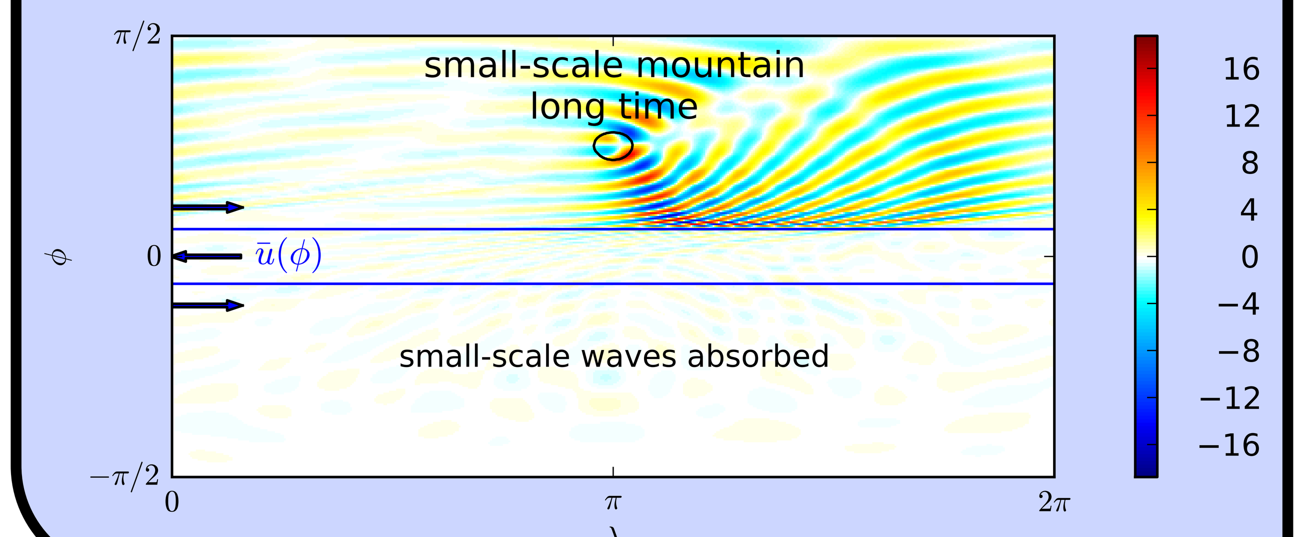
$$\nabla^2\hat{\Psi}(y)e^{im(x-ct)}=\delta(y-c)$$

  - $c\in[0,1]$ with critical points $y_c=c$

Motivation: waves at equator?

- "What waves can penetrate the tradewinds?"
- East-West velocity u for large- and small-scale linear topographic waves





RSW on sphere [7]

- Longitude λ, latitude ϕ, time t
- Wind tangent to sphere surface

$$\vec{u}(\lambda,\phi,t)=u(\lambda,\phi,t)\hat{\lambda}+v(\lambda,\phi,t)\hat{\phi}$$

- Disturbance height $\eta(\lambda,\phi,t)$
- Rossby number $\epsilon\to 0$ for asymptotic limit
- Momentum and height equations

$$\epsilon^2\frac{D\vec{u}}{Dt}-\sin\phi\,\hat{r}\times\vec{u}+\epsilon\nabla\eta=0$$

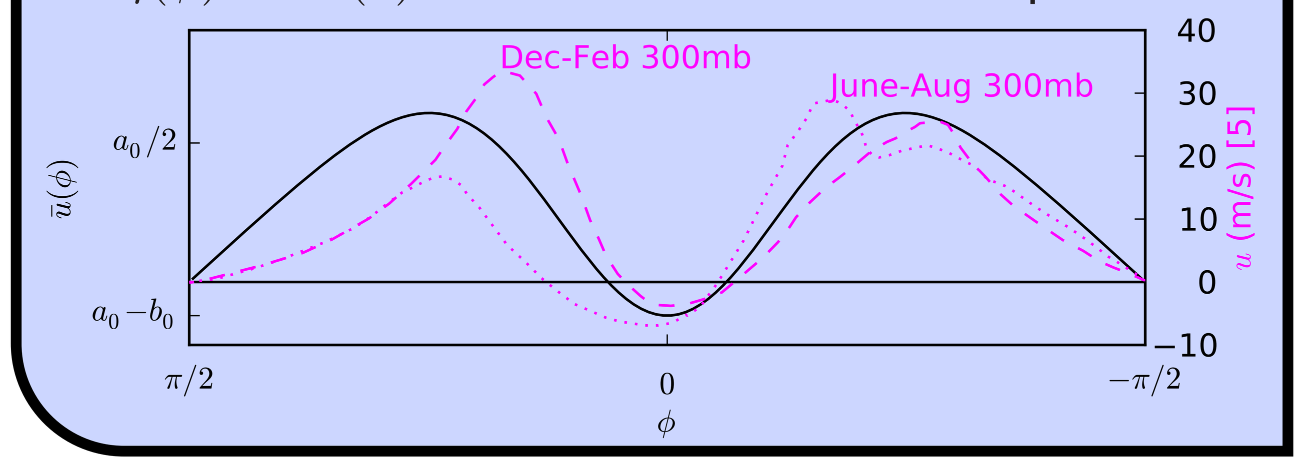
$$\epsilon^2\left(\frac{D\eta}{Dt}+\eta(\nabla\cdot\vec{u})\right)+\epsilon\nabla\cdot\vec{u}=0$$

Background flow

- Latitude-dependent East-West flow

$$\overline{u}(\phi)=a_0\cos\phi-b_0\cos^7\phi \qquad \overline{v}(\phi)=0$$

- Midlatitude jet and equatorial tradewinds
- $\epsilon\overline{\eta}(\phi)=\mathcal{O}(1)$ from RSW momentum equation



RSW eigenvalue problem with background flow

- Background flow plus travelling wave perturbation with phase speed c and wavenumber m

$$\begin{bmatrix}u(\lambda,\phi,t)\\v(\lambda,\phi,t)\\\eta(\lambda,\phi,t)\end{bmatrix}=\begin{bmatrix}\overline{u}(\phi)\\0\\\overline{\eta}(\phi)\end{bmatrix}+\begin{bmatrix}\hat{u}(\phi)\\\hat{v}(\phi)\\\hat{\eta}(\phi)\end{bmatrix}e^{im(\lambda-ct)/\epsilon}$$

- Linearise in $\hat{u},\hat{v},\hat{\eta}$ to obtain an eigenvalue problem in c, which is solved using an FFT method

$$\begin{bmatrix}\epsilon im\left(\frac{\overline{u}}{\cos\phi}-c\right) & -\sin\phi+\epsilon^2\frac{1}{\cos\phi}\left(\cos\phi\,\overline{u}\right)_\phi & \frac{1}{\cos\phi}im\\ \sin\phi+\epsilon^2 2\tan\phi\,\overline{u} & \epsilon im\left(\frac{\overline{u}}{\cos\phi}-c\right) & \epsilon\partial_\phi\\ \frac{1}{\cos\phi}im(1+\epsilon\overline{\eta}) & \epsilon\frac{1}{\cos\phi}\partial_\phi\cos\phi\,(1+\epsilon\overline{\eta}) & \epsilon im\left(\frac{\overline{u}}{\cos\phi}-c\right)\end{bmatrix}\begin{bmatrix}\hat{u}(\phi)\\\hat{v}(\phi)\\\hat{\eta}(\phi)\end{bmatrix}=0$$

- Discrete spectrum (enumerated by n) with no critical latitudes $\Delta_n\equiv\left(\frac{\overline{u}(\phi)}{\cos\phi}\right)_{\min}-c_n>0$
- Literature suggests finite number of discrete modes (maximum n) [3, 6]
- Continuous spectrum with critical latitudes $\frac{\overline{u}(\phi_c)}{\cos\phi_c}-c=0$ associated with equatorial "absorption" [1]

WKB analysis

- Amplitude and fast phase scales

$$\begin{bmatrix}\hat{u}(\phi)\\\hat{v}(\phi)\\\hat{\eta}(\phi)\end{bmatrix}=\begin{bmatrix}U(\phi)\\V(\phi)\\H(\phi)\end{bmatrix}e^{i\rho(\phi)/\epsilon}$$

- $\mathcal{O}(1)$: amplitude null vector with coefficient $A(\phi)$

$$\begin{bmatrix}U\\V\\H\end{bmatrix}=A(\phi)\begin{bmatrix}\rho'\\\frac{m}{\cos\phi}\\-\sin\phi\end{bmatrix}$$

- $\mathcal{O}(\epsilon)$: solvability $\Rightarrow$ eikonal eq. for phase

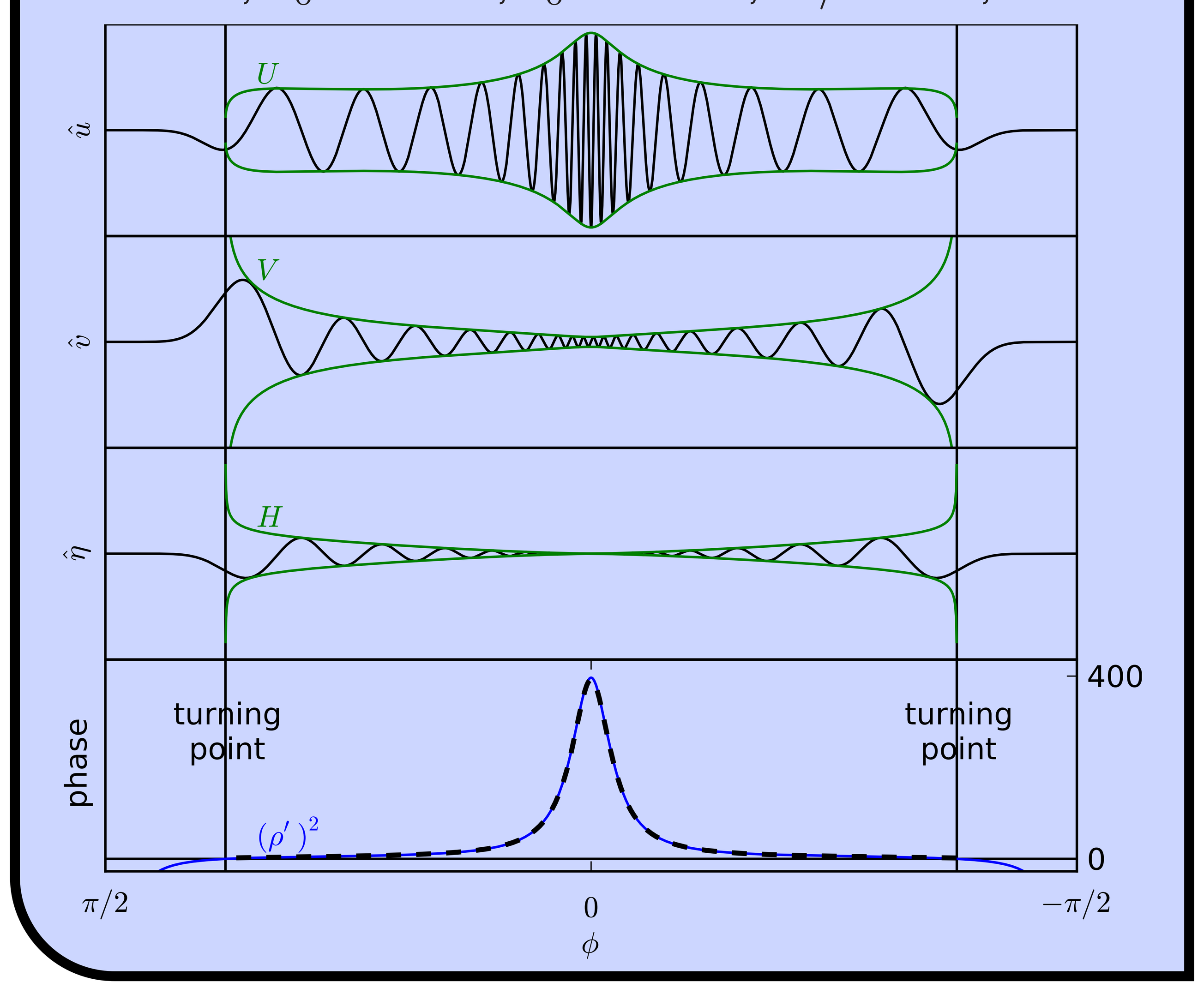
$$\left(\frac{\overline{u}}{\cos\phi}-c\right)\left((\rho')^2+\frac{m^2}{\cos^2\phi}+\frac{\sin^2\phi}{1+\epsilon\overline{\eta}}\right)-\frac{1+\epsilon\overline{\eta}}{\cos\phi}\left(\frac{\sin\phi}{1+\epsilon\overline{\eta}}\right)_\phi=0$$

- eikonal equation has **parallel with Couette**
- $\mathcal{O}(\epsilon^2)$: solvability $\Rightarrow$ transport eq. for $A(\phi)$

$$A(\phi)=\frac{\exp\left(-\frac{1}{2}\int\frac{(\cos\phi\,\overline{u})_\phi+2\sin\phi\,\overline{u}}{\overline{u}-c\cos\phi}d\phi\right)}{\sqrt{\rho'\left(\overline{u}-c\cos\phi\right)(1+\epsilon\overline{\eta})}}$$

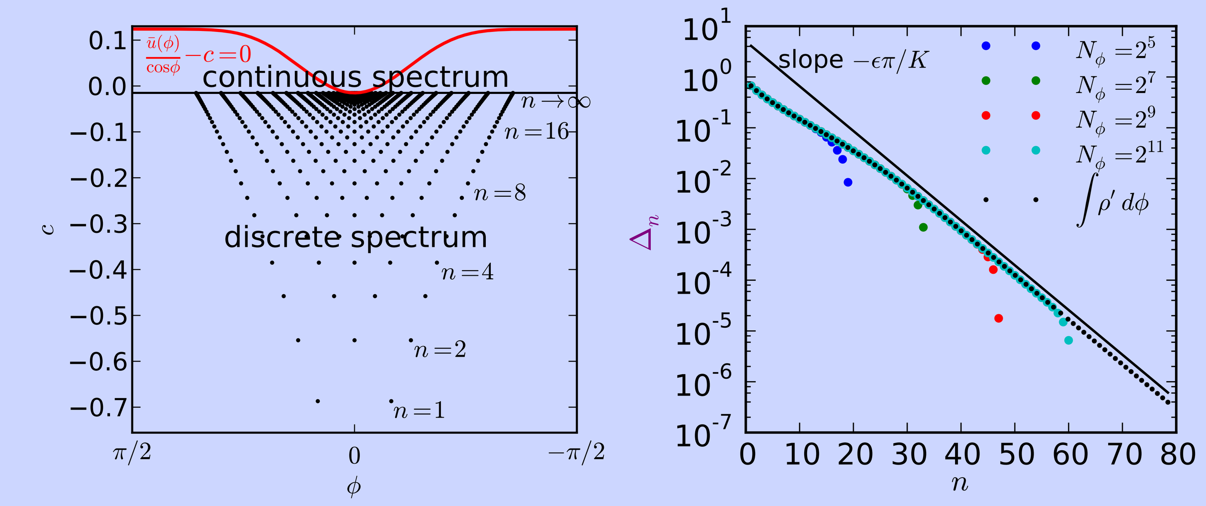
WKB eigenfunction verification

- Compare WKB phase and amplitude to computed eigenfunctions (black)
- $\epsilon=0.1$, $a_0=0.124$, $b_0=0.139$, $m/\epsilon=10$, $n=35$



Discrete spectrum accumulation

- Zerocrossings of the eigenfunctions at $m/\epsilon=10$



- Quantization of phase

$$\epsilon\pi\,(n+1/2)=\int\rho'd\phi$$

- Exponential accumulation in n

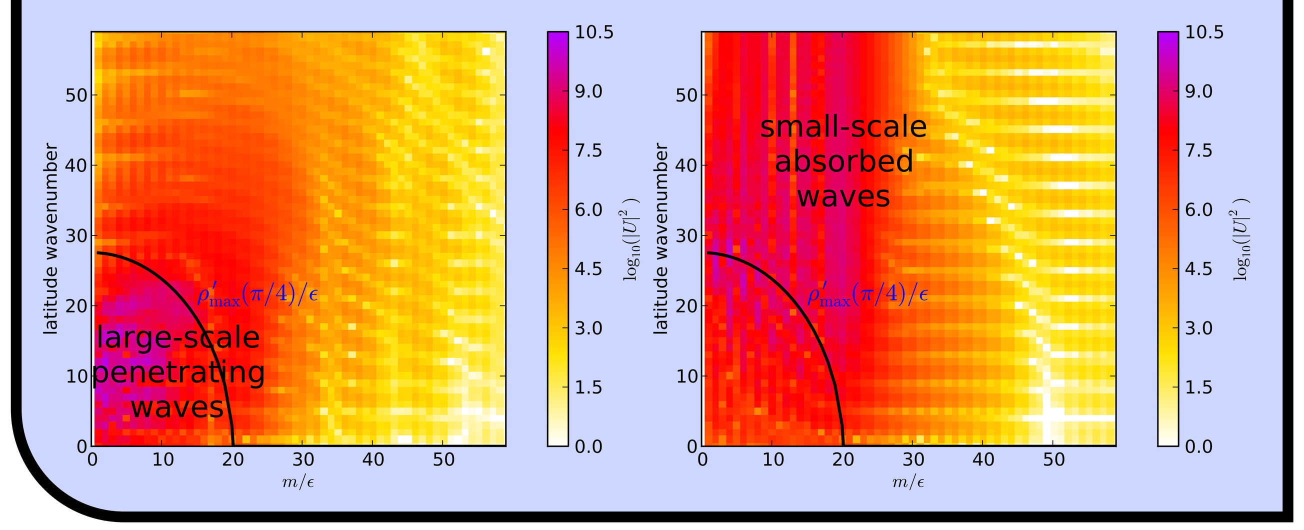
$$\Delta_n\sim\exp\left(-\epsilon\pi Kn\right)\qquad K\equiv\sqrt{\frac{1}{2}\left(\frac{\overline{u}}{\cos\phi}\right)''_{\min}}$$

- Phase speed accumulation point ($\Delta_n\to 0^+$)

$$c_n\to\left(\frac{\overline{u}(\phi)}{\cos\phi}\right)_{\min}\quad\text{as } n\to\infty$$

Penetration by large-scale waves

- FFT of large- and small-scale waves
- Tradewind transparency cutoff $\rho'_{\max}(\pi/4)$
- Scales smaller than cutoff are in continuous spectrum (equatorial absorption)



Conclusions

- Infinite number of discrete spectrum modes
- Long waves in midlatitudes can cross equator
- Use WKB to compute transparency cutoff

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