Unravelling the Resonant Instabilities of a Stratified Gravity Wave

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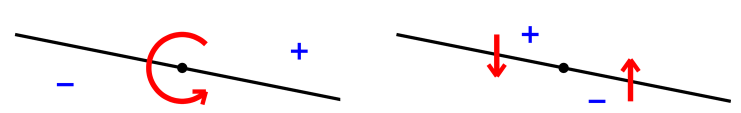
Waves in a 2D Stratified Fluid

Boussinesq Fluid with Constant Stratification

$$\nabla\cdot\vec{\mathbf{u}} = 0 \quad ; \quad \frac{D\eta}{Dt} = -b_x \quad ; \quad \frac{Db}{Dt} = -\mathcal{N}^2 w$$

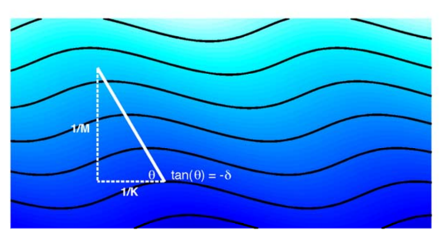


- Streamfunction, $\psi(x,z,t)$ & velocity, $\vec{\mathbf{u}} = (u,w) = (\psi_z, -\psi_x)$
- Buoyancy, $b(x,z,t)$ & vorticity, $\eta(x,z,t) = \psi_{zz} + \psi_{xx}$
- Brunt-Väisälä frequency, $\mathcal{N}$ & stable stratification

Exact Nonlinear Solution — Finite-Amplitude Gravity Wave

$$\begin{pmatrix}\psi\\ b\end{pmatrix} = \begin{pmatrix}-\Omega/KM\\ \mathcal{N}^2/M\end{pmatrix} 2\epsilon\,\sin(Kx+Mz-\Omega t)$$

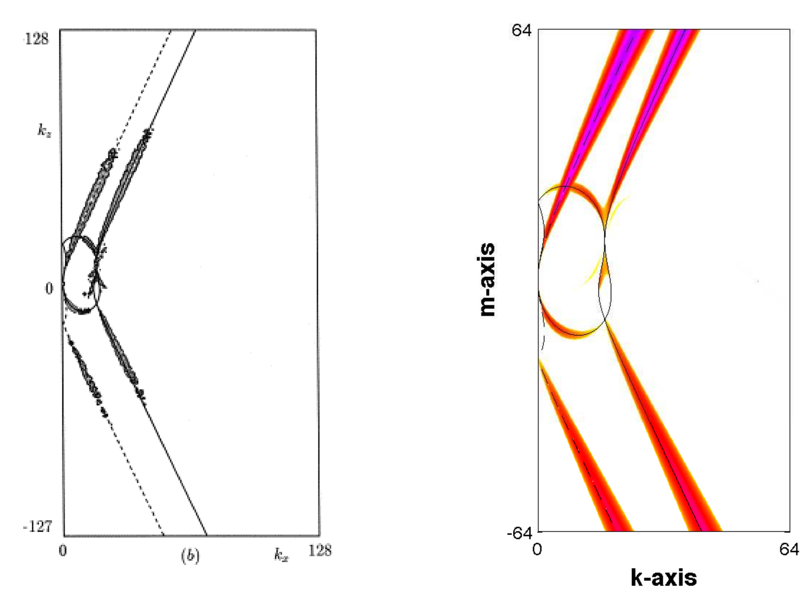
- Primary wavenumbers: (K,M)
- Propagation angle: $\delta = -\dfrac{K}{M}$



- Dispersion relation: $\Omega^2(K,M) = \dfrac{\mathcal{N}^2K^2}{K^2+M^2}$

Unstable Spectrum: DNS & Floquet

- Linear instability with wavenumbers (k,m)
- Growing modes from Direct Numerical Simulations (Lin, 2000) — (left)
- Motivation: unstable eigenvalues using Floquet theory — (right)



References

[1] H. P. McKean & P. van Moerbeke, Inventiones math. 30 (1975)
[2] H. P. McKean & E. Trubowitz, Comm. Pure. and Applied. Math., Vol. XXIX (1976)
[3] R.P. Mied, JFM, 78 (1976)
[4] K. Hasselmann, JFM, 30 (1977).
[5] P.G. Drazin, Proc. Roy. Soc. A, 356 (1977).
[6] M.G. Forest & D.W. McLaughlin, J. Math. Phys 23(7) (1982)
[7] J. Klostermeyer, Geophys. Astrophys. Fluid Dyn., 61 (1991).
[8] L.J. Sonmor & G.P. Klaassen, JFM, 324 (1996).
[9] P.N. Lombard & J.J. Riley, Dyn. Atmos. Oceans, 23 (1996).
[10] C.-L. Lin, Dyn. Atmos. Oceans, 32 (2000).

Linear Stability Analysis

- Linear stability analysis on dimensionless gravity wave + disturbances

$$\begin{pmatrix}\psi\\ b\end{pmatrix} = \begin{pmatrix}-\Omega\\ 1\end{pmatrix} 2\epsilon\sin(x+z-\Omega t) + \begin{pmatrix}\tilde\psi(x,z,t)\\ \tilde b(x,z,t)\end{pmatrix}$$

- Dimensionless frequency: $\Omega^2 = \dfrac{1}{1+\delta^2}$
- Linear PDE system with non-constant, but periodic coefficients

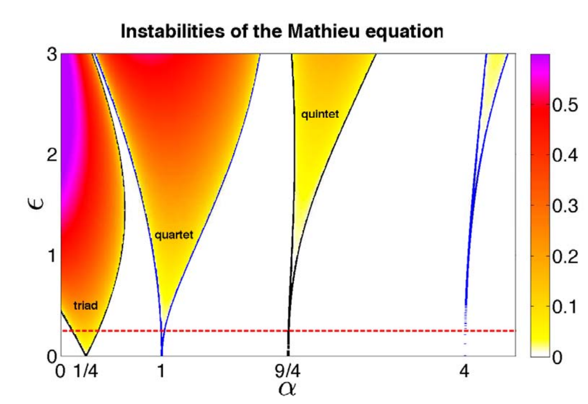
$$\begin{aligned} \tilde\eta_t + \tilde b_x - 2\epsilon J\big(\Omega\tilde\eta + \tilde\psi/\Omega\,,\ \sin(x+z-\Omega t)\big) &= 0\\ \tilde b_t - \tilde\psi_x - 2\epsilon J\big(\Omega\tilde b + \tilde\psi\,,\ \sin(x+z-\Omega t)\big) &= 0\\ \tilde\psi_{zz} + \delta^2\tilde\psi_{xx} &= \tilde\eta \end{aligned}$$

- Linear advection in Jacobian:

$$J(f,g) = \begin{vmatrix} f_x & g_x\\ f_z & g_z\end{vmatrix} = f_x g_z - g_x f_z$$

Floquet Theory

Instabilities of the Mathieu Equation: $\ddot u + (\alpha + \epsilon\sin t)\,u = 0$



- Floquet representation with Fourier series:

$$u(t) = e^{-i\omega t}\left\{\sum_{-\infty}^{+\infty} v_n\, e^{-int}\right\} = \text{exponential part} \times \left\{\text{co-periodic part}\right\}$$

- Floquet exponent, $\omega(\alpha;\epsilon)$ & Im $\omega > 0 \to$ instability

Floquet/Fourier Analysis for PDEs

- Floquet representation with disturbance wavenumbers, (k,m)

$$\begin{pmatrix}\tilde\psi\\ \tilde b\end{pmatrix} = e^{i(kx+mz-\omega t)}\left\{\sum_{-\infty}^{+\infty}\vec v_n\, e^{in(x+z-\Omega t)}\right\}$$

- Floquet exponent, $\omega(k,m;\epsilon)$ & Im $\omega > 0 \to$ instability
- Dispersion relation for disturbances: $\omega^{\pm}(k,m;0) = \pm\dfrac{|k|}{\sqrt{\delta^2k^2+m^2}}$
- A generalized eigenvalue problem with Hill's infinite matrix

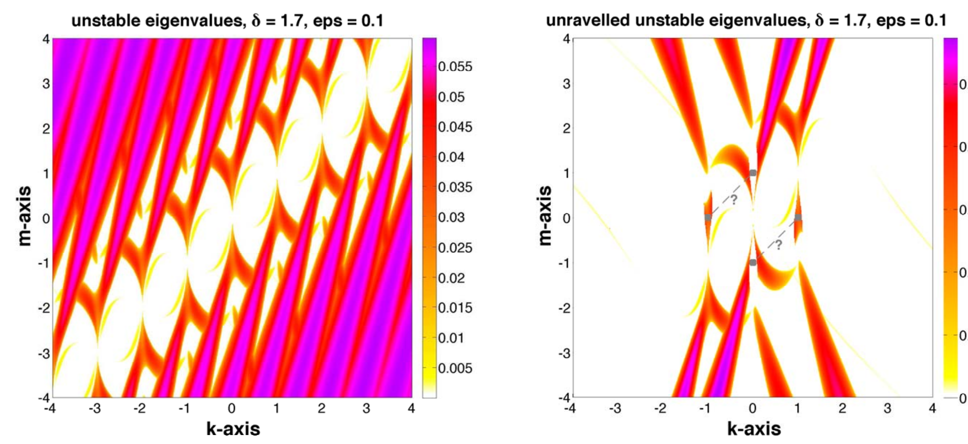
$$\begin{bmatrix} \ddots & \ddots & & \\ \ddots & \mathbf{S}_0 & \epsilon\mathbf{M}_1 & \\ & \epsilon\mathbf{M}_0 & \mathbf{S}_1 & \ddots\\ & & \ddots & \ddots \end{bmatrix} - \omega \begin{bmatrix} \ddots & & & \\ & \mathbf{\Lambda}_0 & & \\ & & \mathbf{\Lambda}_1 & \\ & & & \ddots \end{bmatrix}$$

- $2{\times}2$ $\mathbb{R}$ blocks: $\mathbf{M}_n(k,m)$; $\mathbf{S}_n(k,m)$, symmetric; $\mathbf{\Lambda}_n(k,m)$, diagonal
- Truncate $-N \le n \le N$ & compute $4N{+}2$ eigenvalues $\{\omega(k,m;\epsilon)\}$

A Tangle of Unstable Eigenvalues

- Unstable Floquet eigenvalues selected by maximum growth rate — (left)
- Periodicity from index shifts $\to$ multiple counting of Im ω's

$$\begin{pmatrix}\tilde\psi\\ \tilde b\end{pmatrix} = e^{i((k+q)x+(m+q)z-(\omega+\Omega q)t)}\left\{\sum_{-\infty}^{+\infty}\vec v_{n+q}\, e^{in(x+z-\Omega t)}\right\}$$



- QUESTION 1: is there an association of $\omega(k,m;\epsilon)$ with instabilities corresponding to physical wave resonances — as in Lin (2000)?
- Q1: Yes! An unravelling of unstable Floquet eigenvalues — (right)
- QUESTION 2: are there two values of $\omega(k,m;\epsilon)$ as in the dispersion relation, $\omega^{\pm}(k,m;0)$; or $4N+2$ as from the truncated Hill's matrix?
- QUESTION 3: is the index-periodicity of the Floquet/Fourier method a numerical artifact, or a natural feature of Floquet theory?

Perturbation Analysis

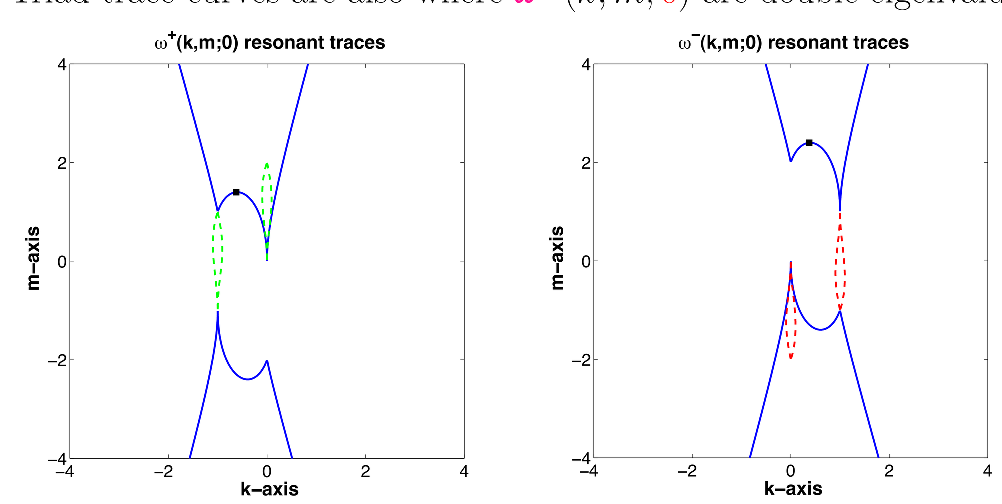
- Two branches from ϵ-perturbation theory: $\omega^{\pm}(k,m;\epsilon) \sim \omega^{\pm}(k,m;0)$
- Complex ω's arise from ϵ-perturbation of multiple Hill's eigenvalues

Triad $(n=\pm1)$ Resonant Trace Curves

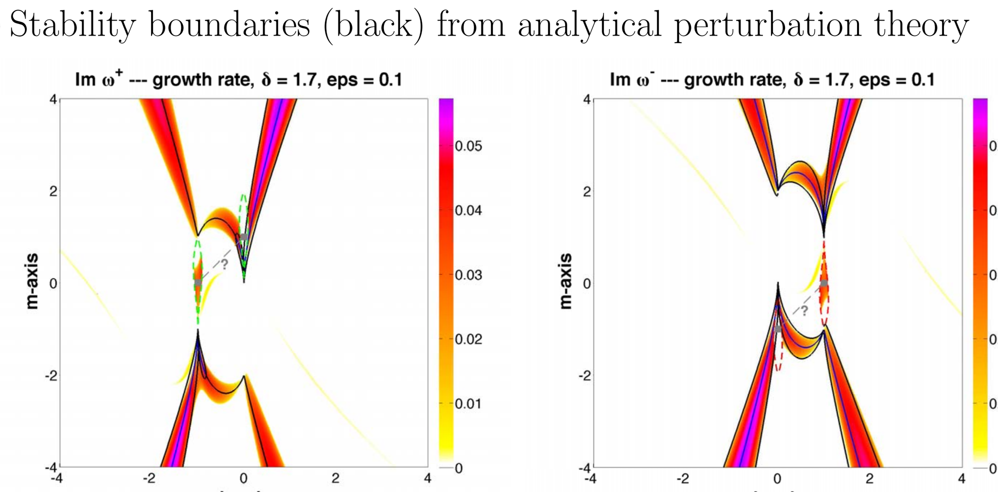
- (k,m) – trace curve for triad resonances: (ω^+, left & ω^-, right)

$$\omega^{\pm}(k,m;0) = \omega^{\pm}(k+n, m+n; 0) - n\Omega$$

- Triad trace curves are also where $\omega^{\pm}(k,m;0)$ are double eigenvalues

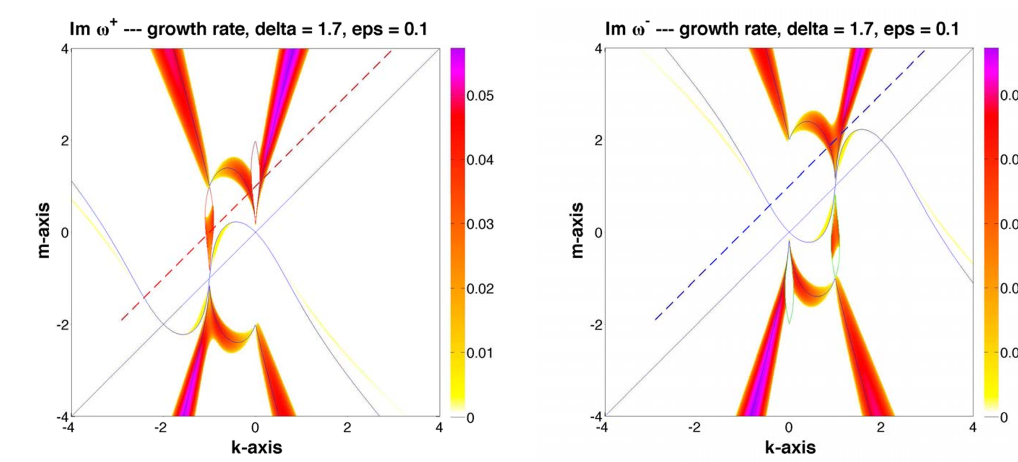


- Triad blue traces are unstable by perturbation theory
- Unravelled Im $\omega^{\pm}$ by ϵ-continuity: *computational perturbation theory*
- Stability boundaries (black) from analytical perturbation theory

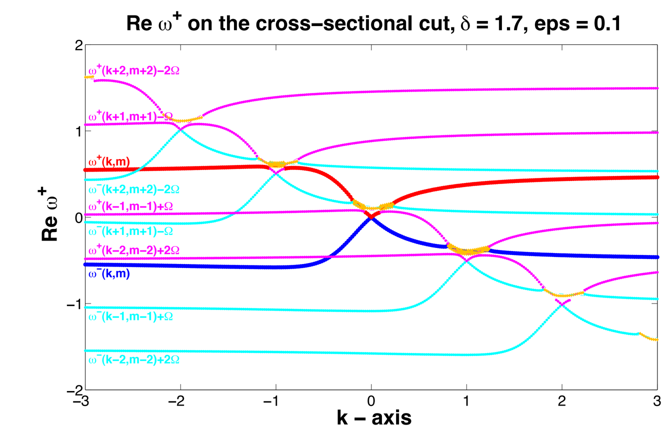


A Riemann Sheet Perspective ?

- Plot 10 curves of Re $\omega^{\pm}$ along the cross-sectional cuts



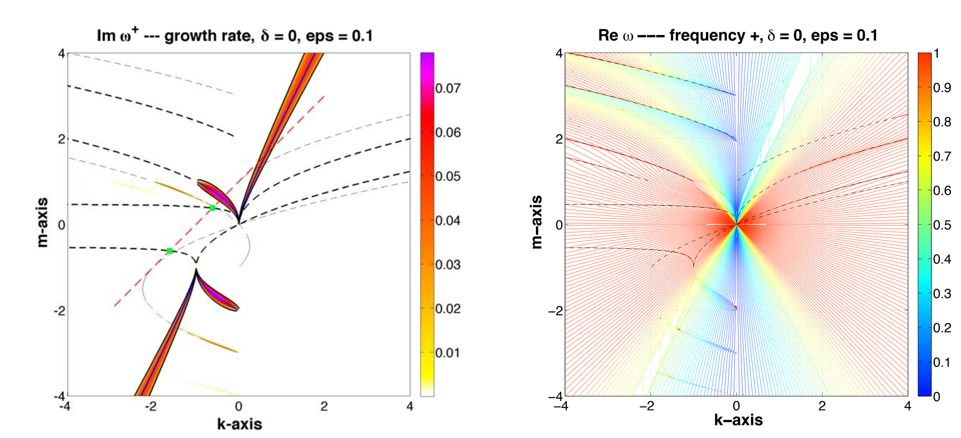
- Unravelled values of Re $\omega^+(k,m)$ & Re $\omega^-(k,m)$
- Other curves correspond to $\omega^{\pm}(k+n, m+n;\epsilon) - n\Omega$



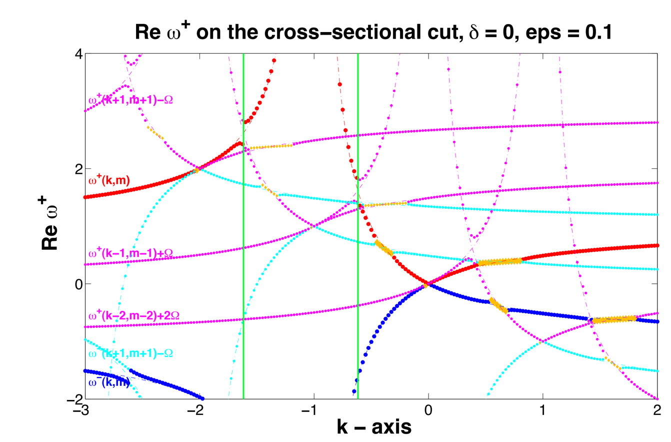
- Layered curves suggest a Riemann sheet interpretation?

Hydrostatic Limit $(\delta = 0)$

- Unravelled Im ω^+ with triad (thick) & quartet (thin) traces — (left)
- Frequency Re ω^+ displays discontinuities across stable traces — (right)



- Discontinuities in Re ω^+ suggest branch cuts in the Riemann sheets



- Connection: do Floquet/Fourier results reflect the Riemann sheets idea of McKean, et al?

Preliminary Conjectures

- Q2: two values of $\omega^{\pm}(k,m;\epsilon)$ are physically important Riemann sheets
- Q2/Q3: $4N+2$ $\omega(k,m;\epsilon)$ reflect truncation of Riemann surface
- ?: complications at high-multiplicity eigenvalues $\to$ crossing traces