Small Rossby Number Corrections to Shallow Water Quasigeostrophy

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Why does rotating shallow water display so little asymmetry in the evolution of balanced vortices at small Rossby number?

In contrast, there is a marked asymmetry of surface QG vortices in a stratified fluid.

Shallow Water Equations

- single-layer, f-plane rotation

$$\mathcal{R}\frac{Du}{Dt} - v = -h_x$$
$$\mathcal{R}\frac{Dv}{Dt} + u = -h_y$$
$$\mathcal{R}\frac{Dh}{Dt} + (\mathcal{B} + \mathcal{R}\,h)(u_x + v_y) = 0$$

- Rossby & Burger numbers

$$\mathcal{R} = \frac{\mathcal{U}}{f\mathcal{L}} \quad ; \quad \mathcal{B} = \frac{g\mathcal{H}}{(f\mathcal{L})^2}$$

- potential vorticity

$$Q = 1 + \mathcal{R}\,q = \frac{1 + \mathcal{R}\,(v_x - u_y)}{1 + \mathcal{R}\,\mathcal{B}^{-1}\,h}$$

- advection of disturbance PV

$$\frac{Dq}{Dt} = 0$$

Shallow Water Potentials: Version I

- a simple extension to quasigeostrophic thinking . . .

$$v = H_x - G$$
$$-u = H_y + F$$
$$h = H - \mathcal{B}\,G_x + \mathcal{B}\,F_y$$

- inversion equations for H, F, G

$$\nabla^2 H - \mathcal{B}^{-1} H = q + \mathcal{R}\,h$$
$$\nabla^2 F - \mathcal{B}^{-1} F = +\frac{\mathcal{R}}{\mathcal{B}}\left\{\left(\frac{Dh}{Dt} + h(u_x+v_y)\right)_x - \frac{Dv}{Dt}\right\}$$
$$\nabla^2 G - \mathcal{B}^{-1} G = +\frac{\mathcal{R}}{\mathcal{B}}\left\{\left(\frac{Dh}{Dt} + h(u_x+v_y)\right)_y + \frac{Du}{Dt}\right\}$$

Small Rossby Number Corrections to QG

- small Rossby number perturbation solution

$$H \sim H^0 + \mathcal{R}\,H^1 + \ldots$$
$$F \sim \mathcal{R}\,F^1 + \ldots$$
$$G \sim \mathcal{R}\,G^1 + \ldots$$

- next-order balanced inversions

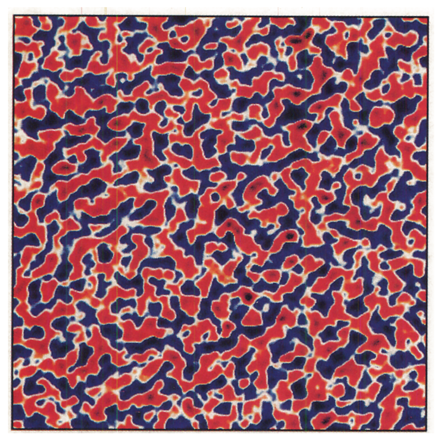
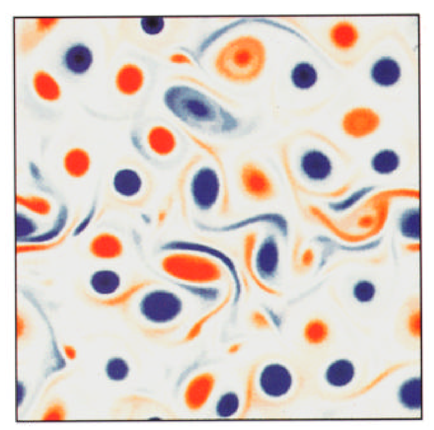
$$\nabla^2 H^0 - \mathcal{B}^{-1} H^0 = q$$
$$\nabla^2 F^1 - \mathcal{B}^{-1} F^1 = -J(H^0, H^0_x)$$
$$\nabla^2 G^1 - \mathcal{B}^{-1} G^1 = -J(H^0, H^0_y)$$
$$\nabla^2 H^1 - \mathcal{B}^{-1} H^1 = q\,H^0$$

- PV dynamics by next-order (swQG^{+1}) winds

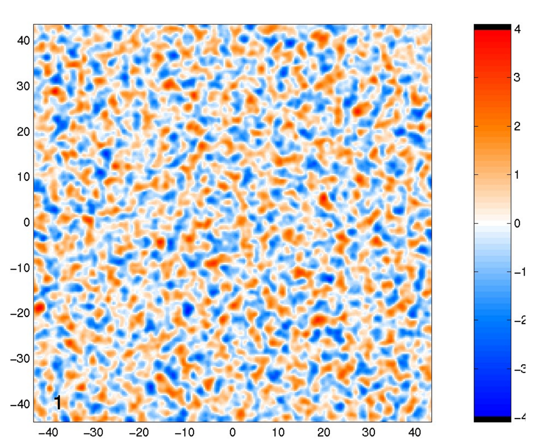
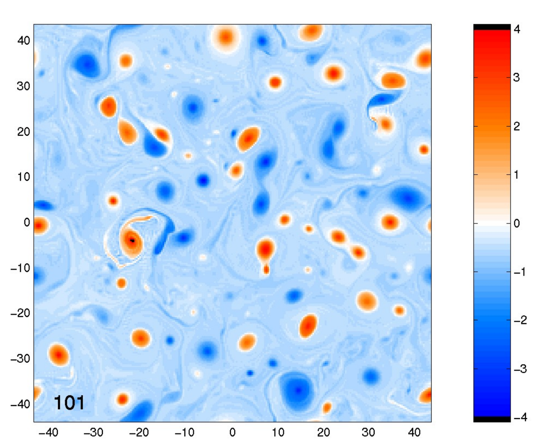
$$\frac{Dq}{Dt} = q_t + (u^0 + \mathcal{R}\,u^1)q_x + (v^0 + \mathcal{R}\,v^1)q_y = 0$$

Freely-Decaying Vortex Dynamics

- Polvani, McWilliams, Spall & Ford (1994)
- near-symmetric dynamics from initial balance
- vorticity evolution in rotating SW with $\mathcal{R} = 0.05$

- Hakim, Muraki & Snyder (2002)
- asymmetric balanced dynamics with $\mathcal{R} = 0.1$
- irreversible cooling of lower surface
- surface potential temperature evolution in sQG^{+1}

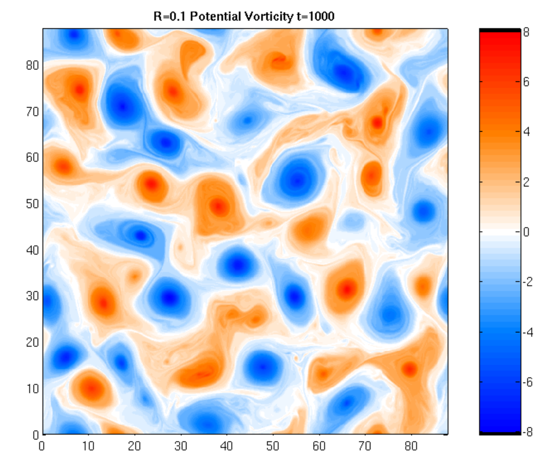
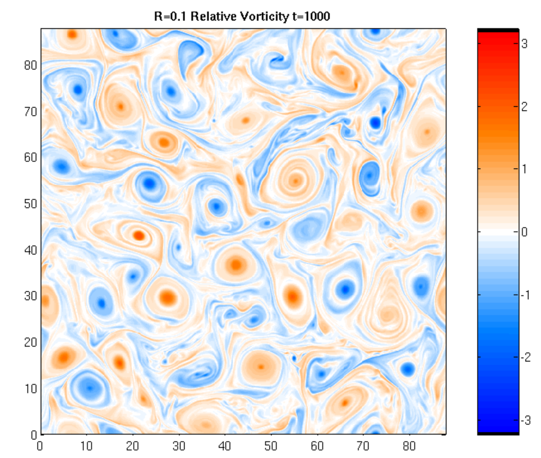



Vortex Dynamics in swQG^{+1}

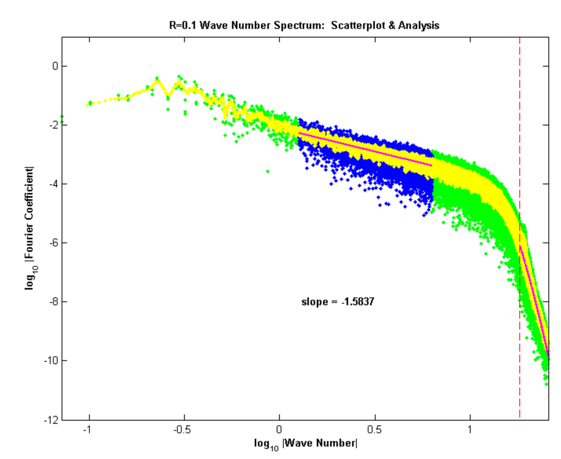
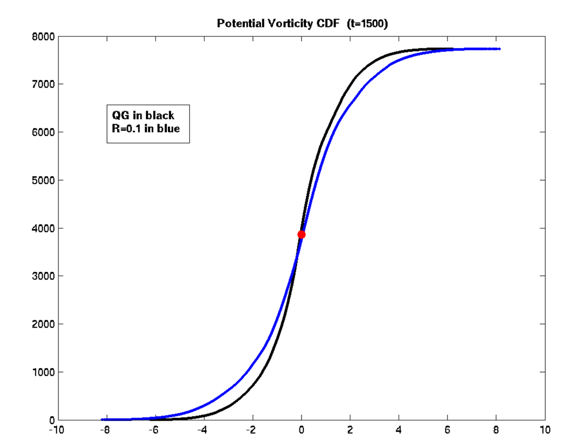
- PV dynamics with ∇^8 hyperdiffusion

$$\frac{Dq}{Dt} = -\nu\,\nabla^8 q$$

- PV & vorticity evolution with $\mathcal{R} = 0.1$

- pseudo-spectral computation with Fourier inversions
- near-symmetry consistent with Polvani, et.al.

Divergent PV Flow, Yet Mean Conserving?

- local convergence/divergence can change mean(PV)

$$q_t + (uq)_x + (vq)_y - (u_x + v_y)\,q = 0$$

- divergence at $O(\mathcal{R})$

$$(\nabla^2 - \mathcal{B}^{-1})\,(u^1_x + v^1_y) = -J(H^0, q)$$

- advection by divergent winds, yet mean(PV) = 0!

$$\iint (u_x + v_y)\,q \sim \mathcal{R}\iint (\nabla^2 - \mathcal{B}^{-1})\,(u^1_x + v^1_y)\,H^0 = 0$$

Oddly, the divergent swQG^{+1} dynamics still conserves mean PV.

This is a significant restriction on the degree of asymmetry which can develop at $O(\mathcal{R})$.

Shallow Water Potentials: Version II

- towards a unifying view for QG & gravity waves . . .

$$v = \psi_x + \phi_x + \chi_y$$
$$-u = \psi_y + \phi_y - \chi_x$$
$$h = \psi + \frac{\mathcal{B}}{Q}\nabla^2\phi$$

- an <u>exact</u> PV streamfunction ($q = 0 \Rightarrow \psi = 0$)

$$\nabla^2\psi - \mathcal{B}^{-1}Q\,\psi = q$$

- inversions via *ageostrophic vorticity* & divergence

$$\gamma = -\nabla^2\left(\frac{\mathcal{B}}{Q}\nabla^2\phi - \phi\right) = v_x - u_y - \nabla^2 h$$
$$\delta = \nabla^2\chi = u_x + v_y$$

- nonlinear wave equations for γ, δ

$$\mathcal{R}\frac{D\gamma}{Dt} - \left[\nabla^2 - \frac{Q}{\mathcal{B}}\right]((\mathcal{B} + \mathcal{R}\,h)\delta)$$
$$= \mathcal{R}\left\{-u(\nabla^2 h)_x - v(\nabla^2 h)_y + \nabla^2(uh_x + vh_y)\right\}$$

$$\mathcal{R}\frac{D\delta}{Dt} - \gamma$$
$$= \mathcal{R}\left\{2J(u,v) - \delta^2\right\}$$

Gravity Wave Dynamics

- linearized wave equations (Q=1)

$$\mathcal{R}\,\gamma_t - \mathcal{B}\,\nabla^2\delta + \delta = 0$$
$$\mathcal{R}\,\delta_t - \gamma = 0$$

- dispersion relation

$$\mathcal{R}^2\omega^2 = \mathcal{B}\,(k^2 + l^2) + 1$$

- exact nonlinear fast manifold with q=0

$$\mathcal{R}\frac{Du}{Dt} - v = -\mathcal{B}\,(v_x - u_y)_x$$
$$\mathcal{R}\frac{Dv}{Dt} + u = -\mathcal{B}\,(v_x - u_y)_y$$

Gravity Wave Generation by PV Dynamics

- PV forcing terms for wave generation