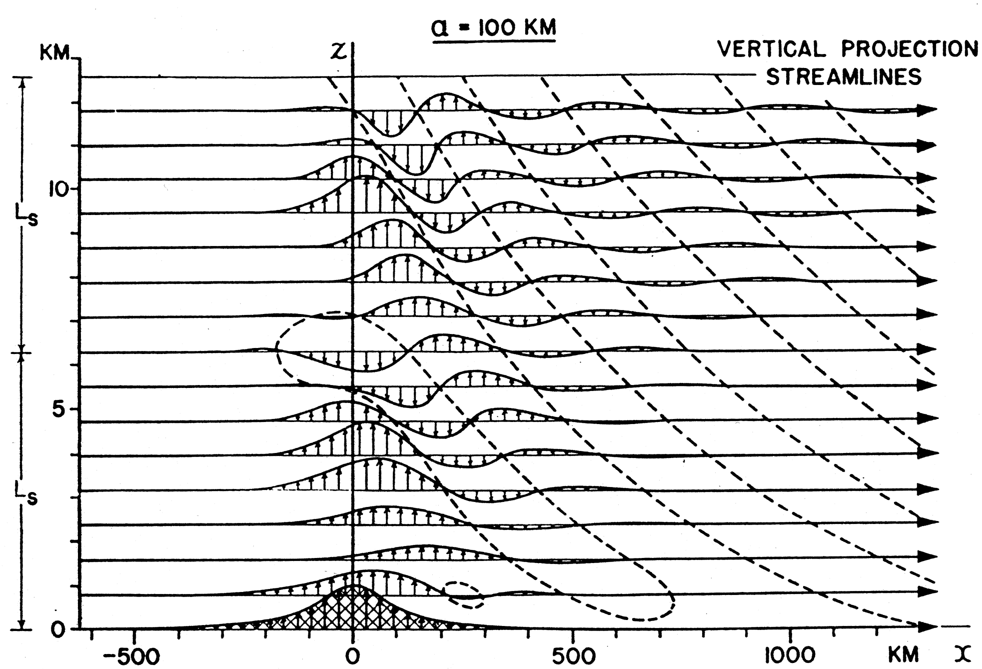
Gravity Waves with Topography . . .
. . . and Possibly Without?

- ▷ stratified, rotating & hydrostatic flow
- ▷ revisiting Queney's flow over a mesoscale ridge
- ▷ balance, waves & applied mathematics



Queney, 1947

- ▷ Dave Muraki (Simon Fraser University)
- ▷ Chris Snyder & Rich Rotunno (NCAR Boulder)

Queney's Displacement Streamlines

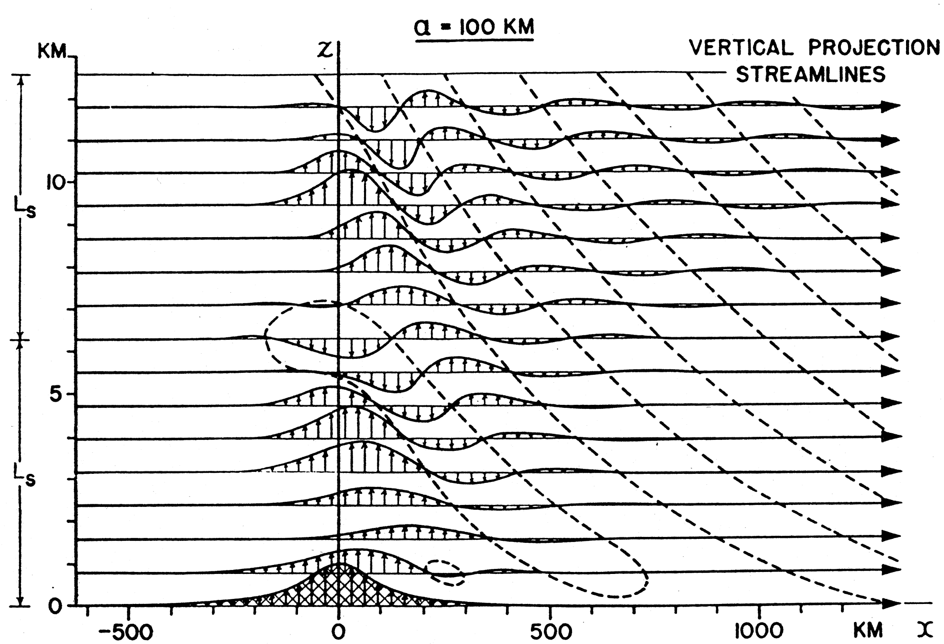
Flow over a 2D Mesoscale Ridge

- Queney 1947, 1948; Smith 1979; Gill 1982
- vertical displacement from buoyancy anomaly $b(x, z)$

$$z(x) = z^{\infty} - \frac{1}{N^2}\, b(x, z^{\infty})$$

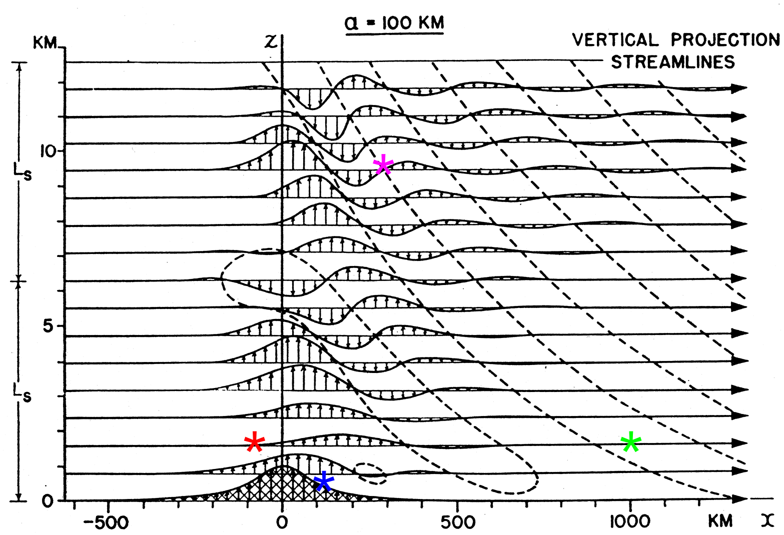
- rotating & hydrostatic case: parameters

$$\mathcal{R} = \frac{U}{fL} = 1 \qquad ; \qquad \mathcal{F} = \frac{U}{NH} = 1$$

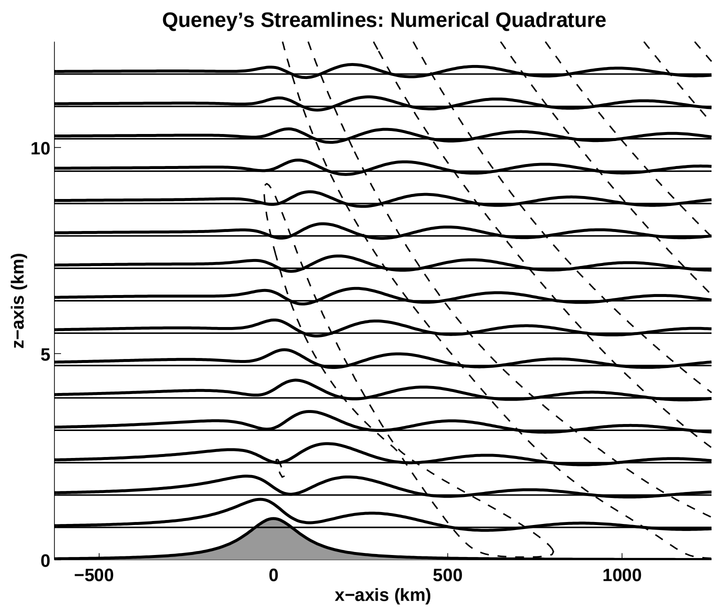


Queney, 1947

Displacements Recomputed



Queney, 1947



Muraki, 2001

Comparison

Missing Features in Queney 1947

- ▷ windward maxima of upward displacement (low level) *
  - → as in non-rotating case
- ▷ organized downdraft into downslope windstorm *
- ▷ convergence of (low level) streamlines in lee *
  - → as consistent with pressure drag in non-rotating case
- ▷ persistence of low level waves downstream *
  - → as in surface analysis of (Pierrehumbert 1984)
- ▷ upward mean vertical displacement of far-field waves *
  - → as in QG theory

A Tale of Two Fourier Calculations

- ▷ Queney's calculation based on *stationary-phase* approximation
  - → problematic at surface, summit & ridge zenith
- ▷ direct numerical quadrature of Fourier integral
  - → singularly-oscillating integrand gives aliasing errors
  - → we resolve using desingularized quadrature

Queney's Linear Theory (1947)

Rotating Case

- Fourier integral for buoyancy anomaly

$$b(x,z) = -\frac{N^2}{\pi} \text{Real} \left\{ \int_0^\infty \hat{h}(k)\; e^{ikx}\; e^{im(k)z}\; dk \right\}$$

- bell-shaped topography & Fourier transform

$$h(x) = \frac{HL^2}{L^2 + x^2} \quad ; \quad \hat{h}(k) = \pi H L\, e^{-|k|L}$$

- inertial wavenumber (k_f) & Scorer parameter (k_s)

$$k_f = \frac{f}{U} \quad ; \quad k_s = \frac{N}{U}$$

incident wind U, f-plane Coriolis, stratification N

- 2D linear dispersion relation with rotation

$$m(k) = \begin{cases} ik_s \dfrac{k}{\sqrt{{k_f}^2 - k^2}} & \text{for} \quad 0 \le k < k_f \\ k_s \dfrac{k}{\sqrt{k^2 - {k_f}^2}} & \text{for} \quad k_f < k < \infty \end{cases}$$

small $k \longrightarrow$ vertical decay; large $k \longrightarrow$ outgoing waves

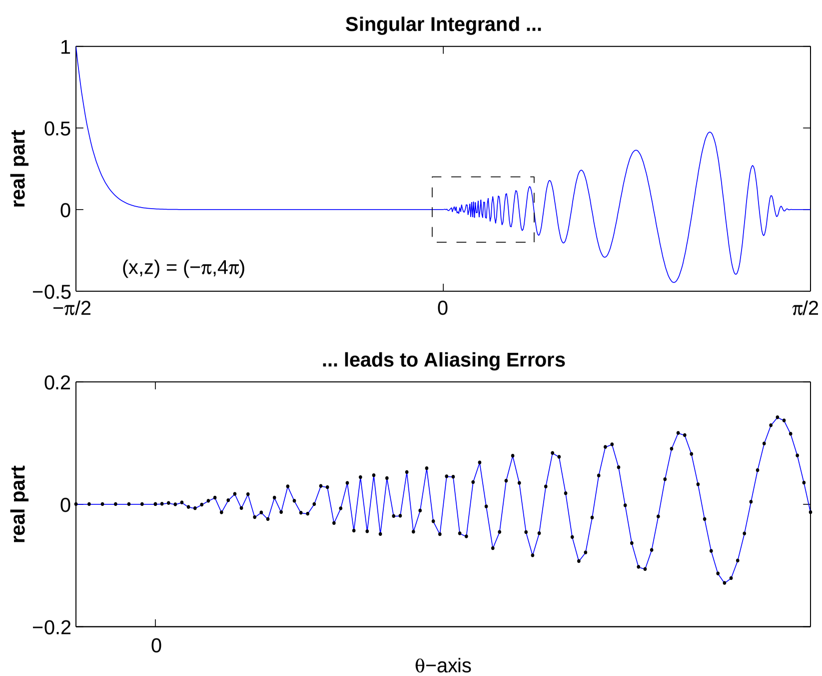
Desingularization I

Singular Exponent

- ▷ vertical wavenumber $m(k) \to \infty$, as $k \to k_f^+$
- ▷ Queney's trigonometric coordinates

$$k = \begin{cases} -k_f \sin\theta & \text{for} \quad -\frac{\pi}{2} \le \theta \le 0 \quad \text{(decay)} \\ k_f \sec\theta & \text{for} \quad 0 < \theta < \frac{\pi}{2} \quad \text{(waves)} \end{cases}$$

- ▷ amplitude of integrand $\to 0$, as $\theta \to 0^+$

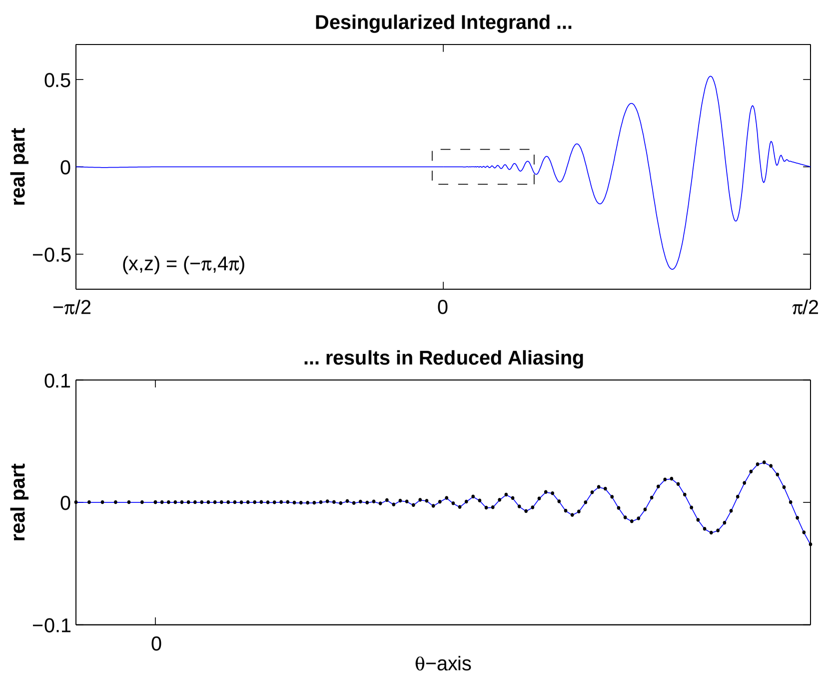


Desingularization II

Numerical Errors

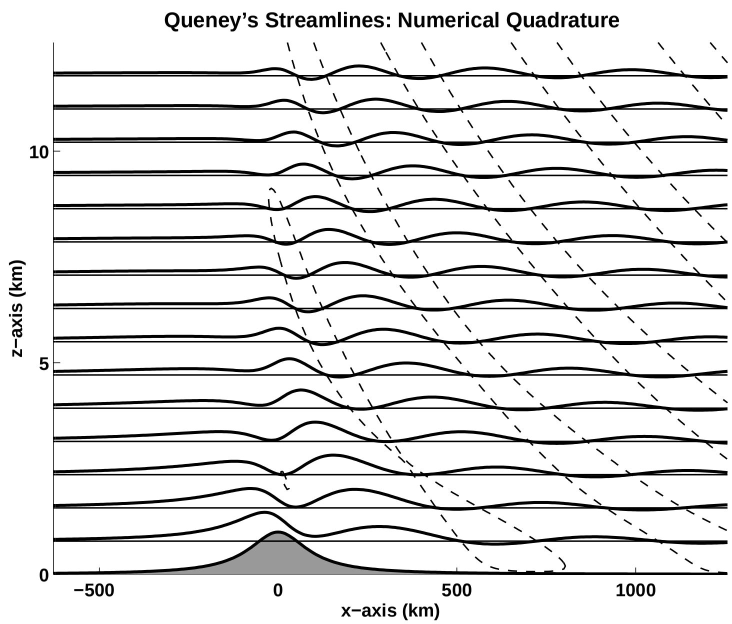
- ▷ FFT-based quadratures produce false periodicity
- ▷ aliasing errors $\longrightarrow$ upstream wavy artifacts & downstream interference
- ▷ evaluate singular $\mathcal{E}_n$-integrals using *exponential integral*, $Ei(x)$

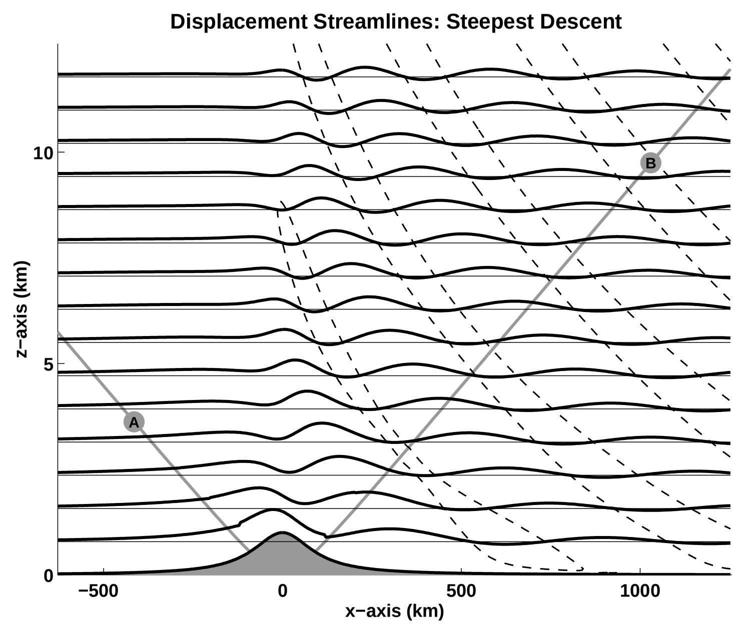
$$\mathcal{E}_n = \int_0^{\pi/2} e^{ik_S z \csc\theta} \sin^n\theta \, \cos\theta \, d\theta$$



- ▷ analogous problem as $\theta \longrightarrow \pi/2$ for large $\mathcal{R}$(Smith 82)

Steepest Descent Approximation





Singular Asymptotic Approach

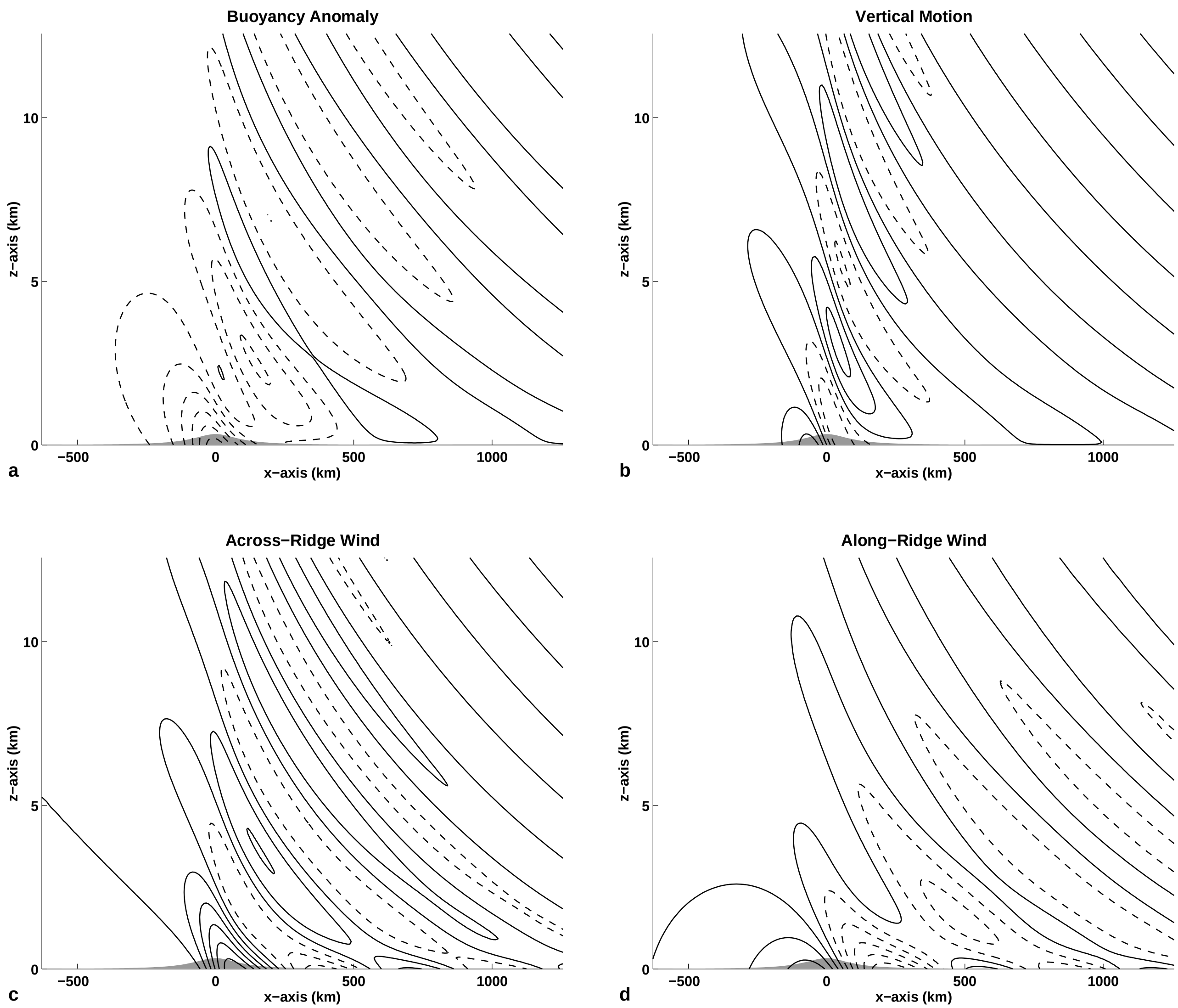
- ▷ complex-analytic method
- ▷ decay of wave amplitude in zenith (no e-folding height)

$$\text{wave amplitude} \;\propto\; (\mathcal{R}z)^{1/6}\ \exp\left\{-C\,\mathcal{R}^{-2/3}z^{1/3}\right\}$$

- ▷ surface-scattered wave below **B**-line

Other Fields

- ▷ desingularized quadratures for vertical motion & disturbance winds ($\mathcal{R} = 1.0$, $\mathcal{F} = 3.0$)



3D Topography

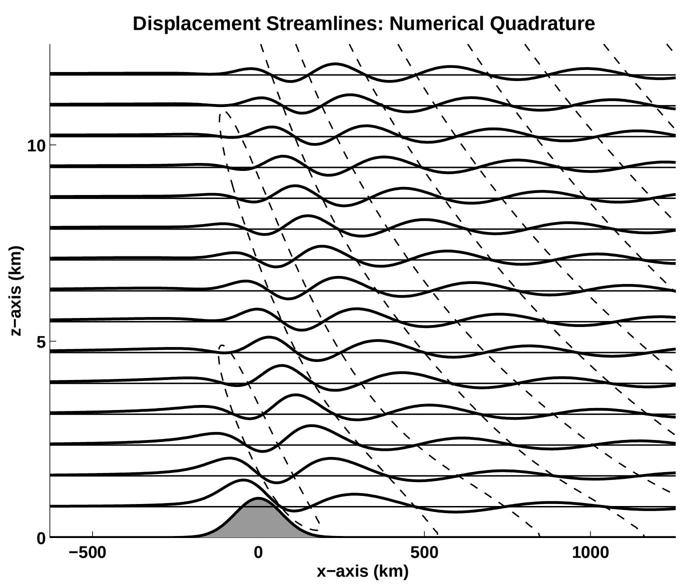
Flow Past a Circular Gaussian Mountain ($\mathcal{R} = 1$, $\mathcal{F} = 1$)

- 3D linear dispersion relation

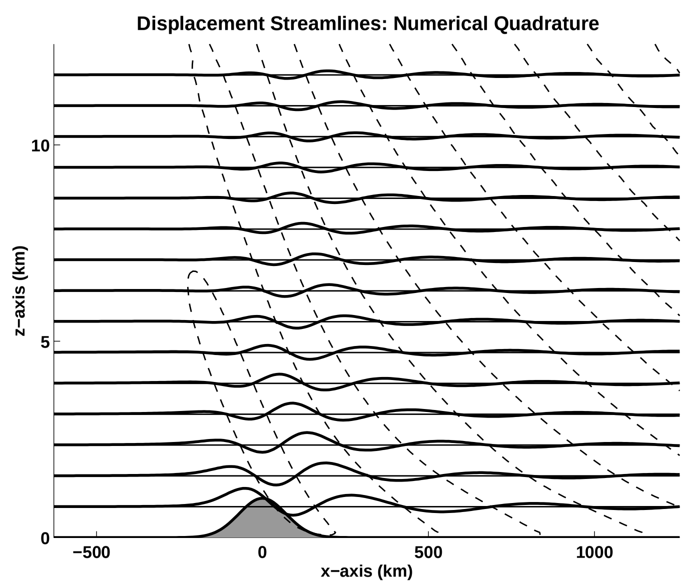
$$m(k,l) = \begin{cases} i k_s \sqrt{\dfrac{k^2 + l^2}{{k_f}^2 - k^2}} & \text{for} \quad 0 \le k < k_f \\ k_s \sqrt{\dfrac{k^2 + l^2}{k^2 - {k_f}^2}} & \text{for} \quad k_f < k < \infty \end{cases}$$

- same desingularization integrals apply

2D gaussian ridge



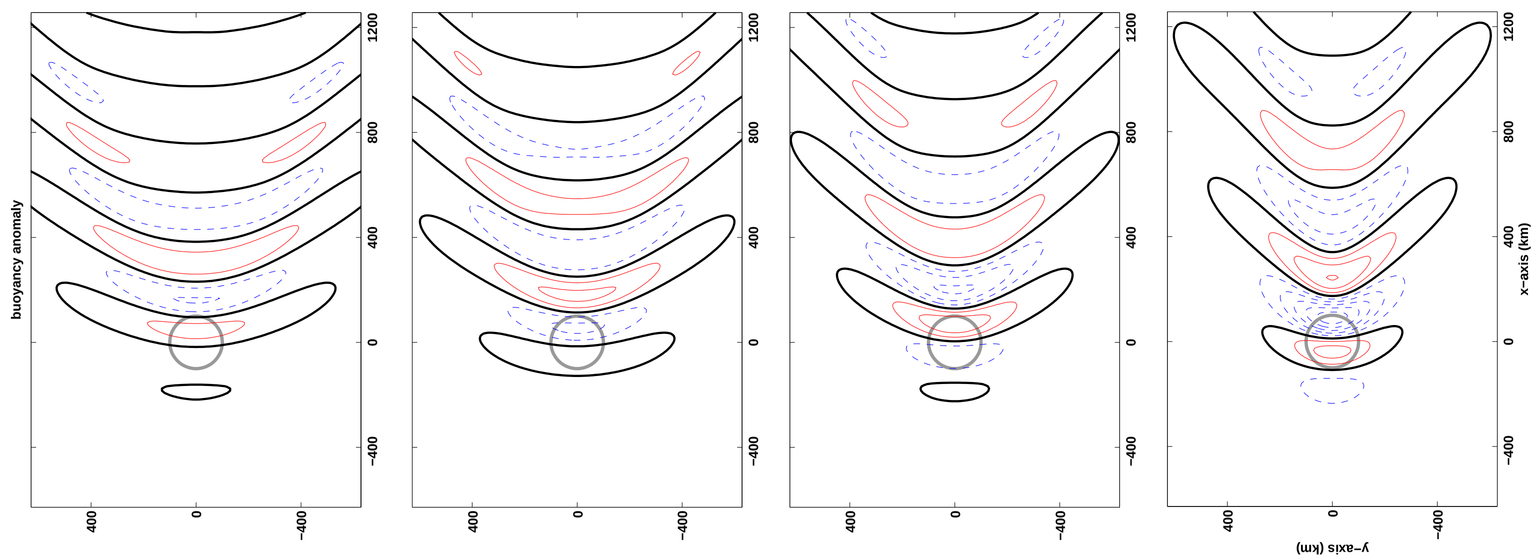
3D gaussian mountain



Circular Mountain

- ▷ buoyancy anomaly: $\mathcal{R} = 1$, $\mathcal{F} = 1$
- ▷ windward tilt of phase
- ▷ wake dispersion downstream

$z = 2\pi$ $z = 3\pi/2$ $z = \pi$ $z = \pi/2$



- ▷ topographic study with Epifanio & Rotunno

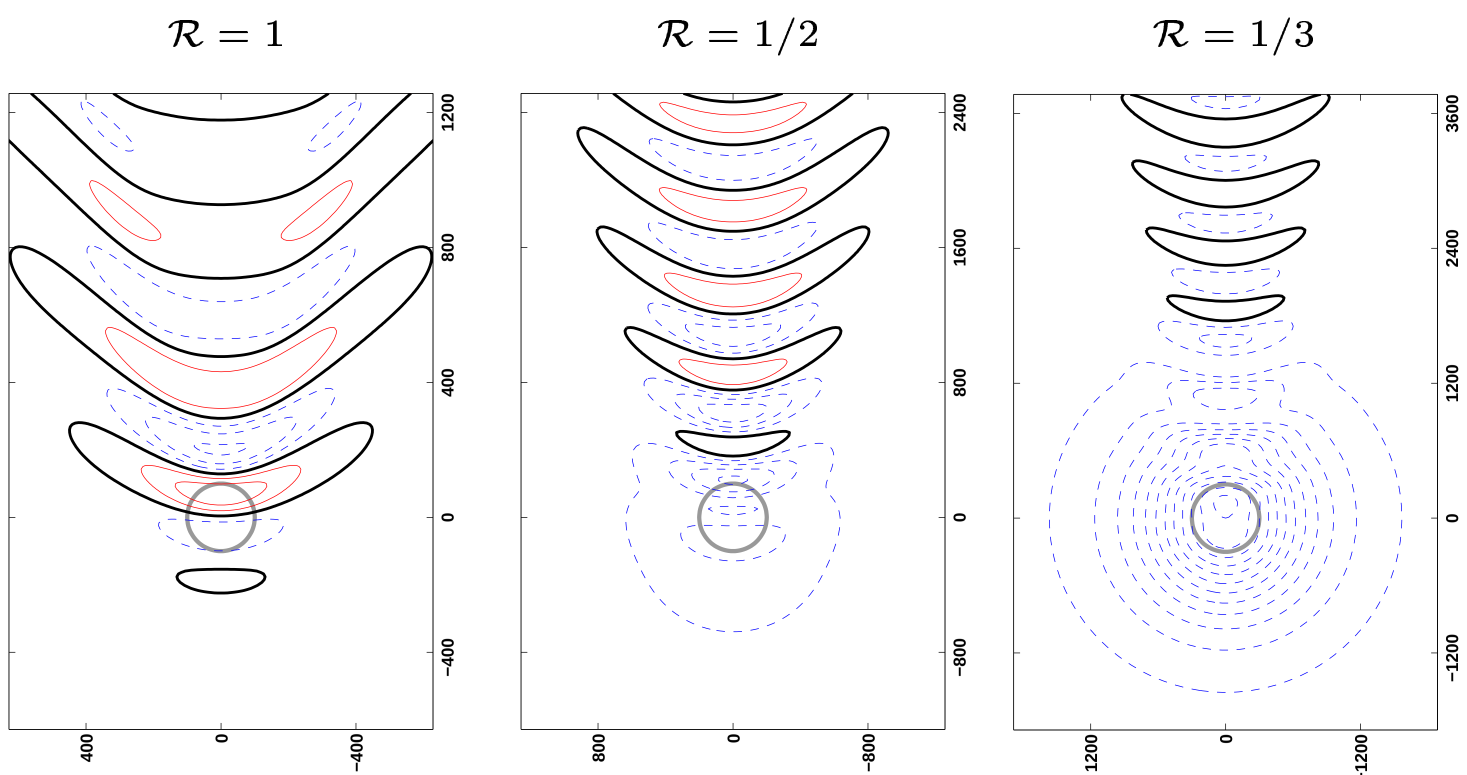
Transition to QG

- buoyancy anomaly at $z = \pi$: $\mathcal{F} = 1$
- $\mathcal{R} \to 0$: by decreasing incident wind

  $\to$ development of QG anticyclone

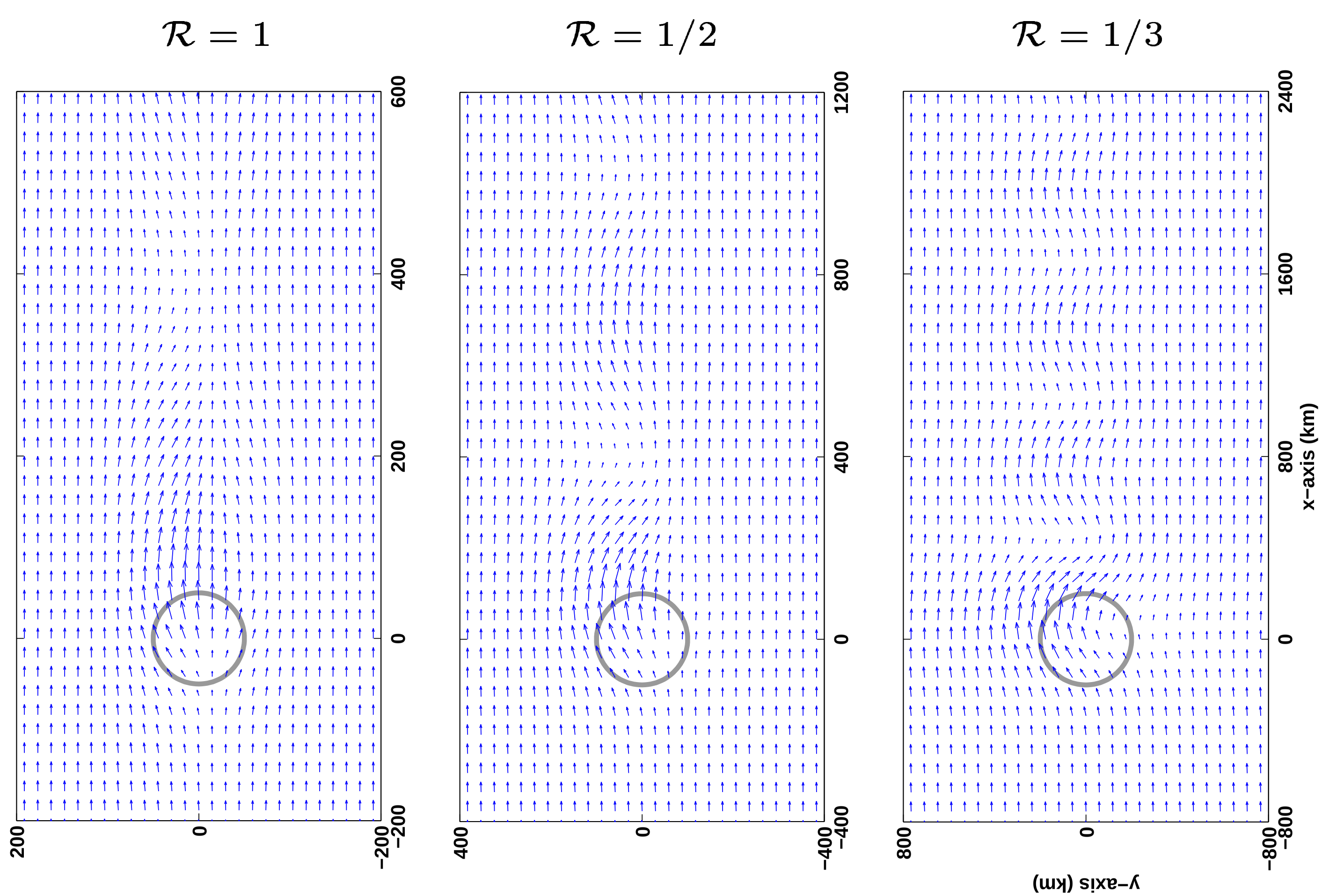
  $\to$ wavelength decreases like $\mathcal{R}$

  $\to$ wave amplitude decreases $\approx e^{-1/\mathcal{R}}$?? (decreased contour interval)



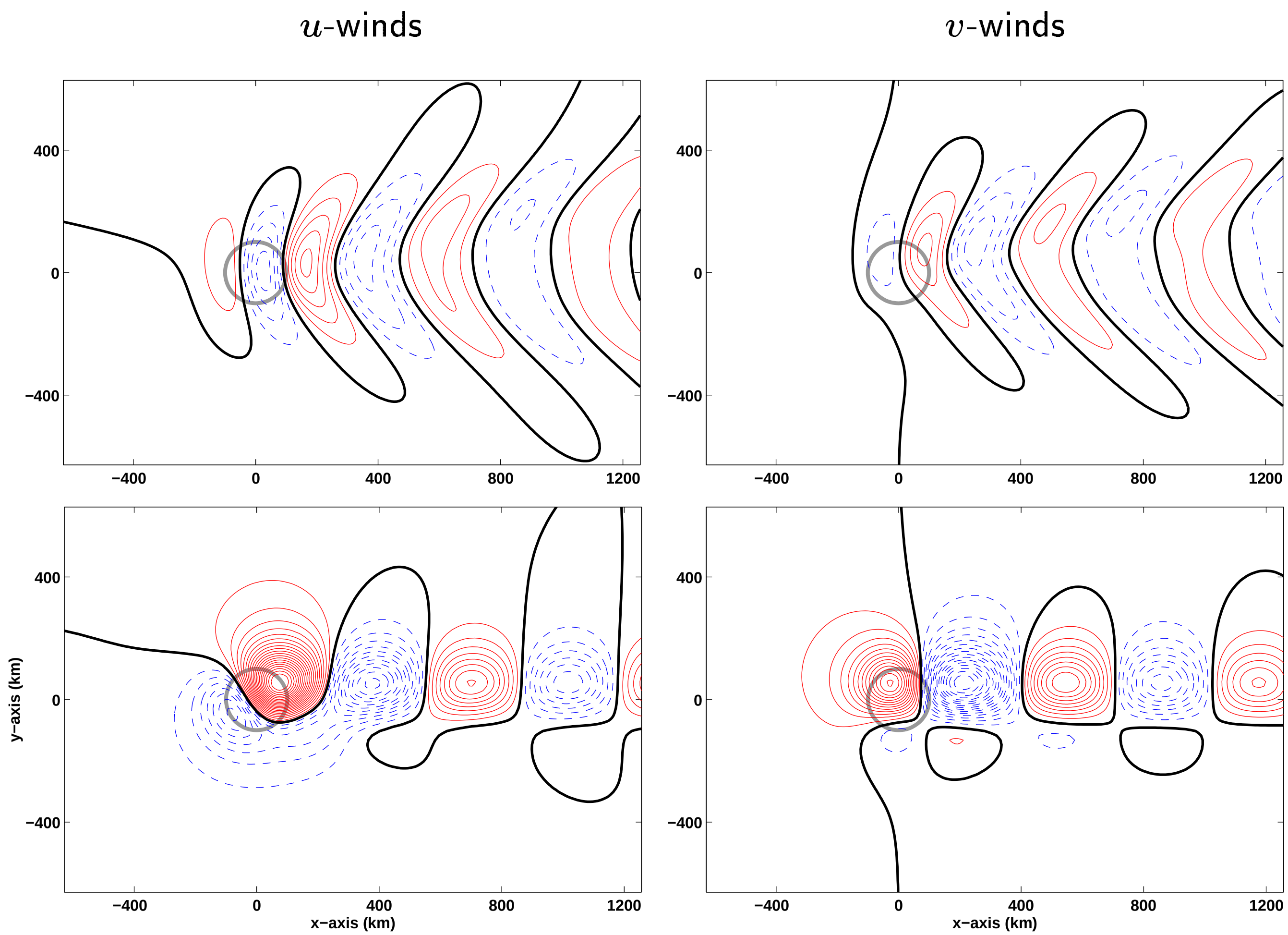
Transition to QG

- surface wind vectors: $\mathcal{F} = 1$
- transition from split flow to anticyclone as $\mathcal{R} \to 0$



Disturbance Winds

- $\mathcal{R} = 1$, $\mathcal{F} = 1$ at heights $z = \pi/2, 0$ km

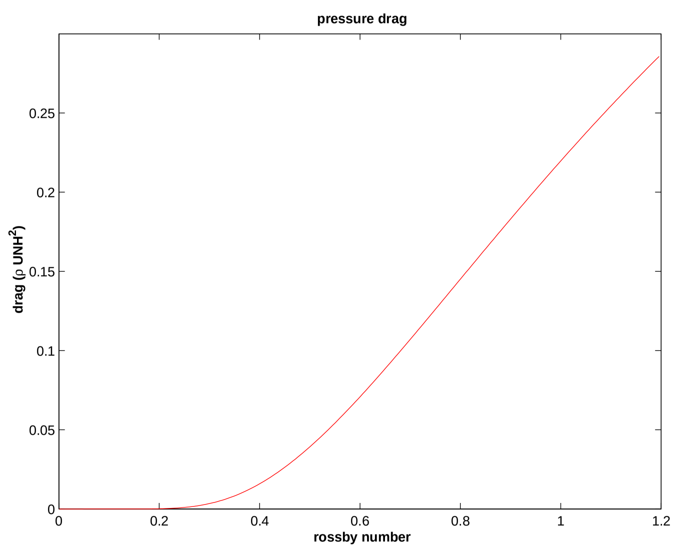


Waves & the Rossby Number

Flow Over Topography

- QG theory: cap vortex only
- Queney theory: $\mathcal{R}$-transition from QG to waves
- pressure drag as indicator of wave generation (2D ridge)

$$\mathrm{drag} \sim \frac{1}{\mathcal{R}}\, e^{-1/\mathcal{R}}$$



- transition-like exponential increase in wave action

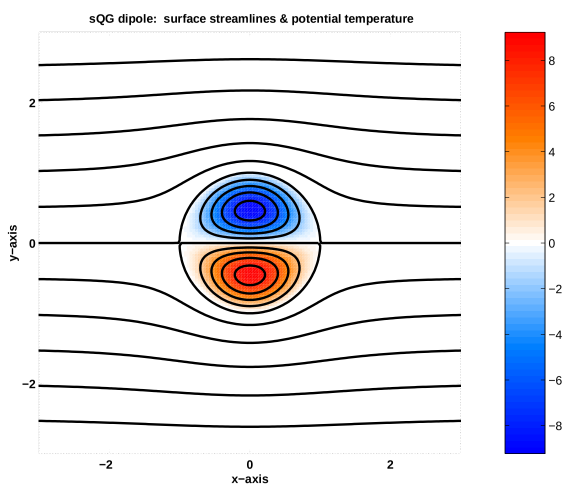
Wave Generation

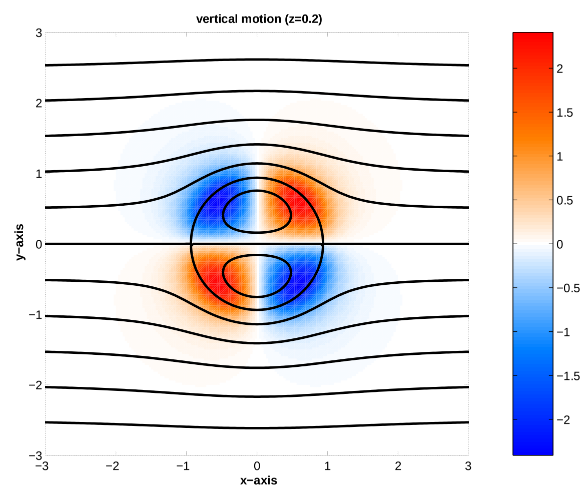
A General Mechanism

- ▷ lessons from weak topography:
  - $\longrightarrow$ wind moving past vertical motion produces a gravity wave wake
  - $\longrightarrow$ waves are exponentially small when $\mathcal{R}$ small

Wave Generation by a *Balanced* Flow?

- ▷ QG dynamics induce vertical motion via <u>omega equation</u>
- ▷ question: wave generation as finite-$\mathcal{R}$ correction to QG flow?
- ▷ experiment: gravity wave wake from a moving sQG dipole





Quasigeostrophy & Vertical Motion

Rotating, Stratified Flow

- zero Rossby number limit of primitive equations (PE)
- geostrophy: representation of all variables by a potential ϕ

$$\begin{aligned} v &= \phi_x \\ -u &= \phi_y \\ \theta &= \phi_z \end{aligned}$$

- quasigeostrophy: PV dynamics & inversion

$$\frac{Dq}{Dt} = 0 \qquad ; \qquad \nabla^2 \phi = q \qquad (+\ \text{BCs})$$

- omega equation: vertical motion from $\vec{Q}$-vector

$$\nabla^2 w = \nabla \cdot \vec{Q} \qquad (+\ \text{BCs})$$

- gravity waves purged by single time-derivative construction

sQG Dipole

- zero interior PV ($q = 0$), surface θ dynamics
- steady, translating vortex pair solution

Beyond Quasigeostrophy

Three-Potential Representation

- exact reformulation of PE for finite Rossby number (QG$^+$)
- winds & vertical motion

$$\begin{pmatrix} u \\ v \\ \mathcal{R}w \end{pmatrix} = -\nabla \times \begin{pmatrix} G \\ -F \\ \Phi \end{pmatrix} = \begin{pmatrix} -\Phi_y & -F_z & \\ +\Phi_x & & -G_z \\ & +F_x & +G_y \end{pmatrix}$$

- potential temperature

$$\theta = \nabla \cdot \begin{pmatrix} G \\ -F \\ \Phi \end{pmatrix} = G_x - F_y + \Phi_z$$

- PV dynamics

$$\frac{Dq}{Dt} = q_t + u\, q_x + v\, q_y + \mathcal{R}\, w\, q_z = 0$$

- three inversions

$$\begin{aligned} \nabla^2\Phi &= q + \mathcal{R}\left((v_x - u_y)\,\theta_z - v_z\theta_x + u_z\theta_y\right) \\ \nabla^2 F &= \mathcal{R}\left(-\left(\frac{D\theta}{Dt}\right)_x + \left(\frac{Dv}{Dt}\right)_z\right) \quad \rightarrow \quad \vec{Q}^x \\ \nabla^2 G &= \mathcal{R}\left(-\left(\frac{D\theta}{Dt}\right)_y - \left(\frac{Dy}{Dt}\right)_z\right) \quad \rightarrow \quad \vec{Q}^y \end{aligned}$$

- plus boundary conditions . . .

A Model for Wave Generation

F, G Correction Potentials

- small, but finite Rossby number ($\mathcal{R} \approx 0.4$)
- diagnostic F, G-equations with incident wind U^∞

$$\nabla^2 F + \mathcal{R}\left(\frac{\partial}{\partial t} + U^\infty \frac{\partial}{\partial x}\right)(G_{xx} - F_{xy} + G_{zz}) \quad \sim \quad \vec{Q}^x \;=\; 2\,\mathcal{R}\,J(\Phi_z, \Phi_x)$$

$$\nabla^2 G + \mathcal{R}\left(\frac{\partial}{\partial t} + U^\infty \frac{\partial}{\partial x}\right)(G_{xy} - F_{yy} - F_{zz}) \quad \sim \quad \vec{Q}^y \;=\; 2\,\mathcal{R}\,J(\Phi_z, \Phi_y)$$

- surface BCs ($z = 0$):

$$F_x + G_y = \mathcal{R}\,w \qquad ; \qquad G_x - F_y = 0$$

Coupling Equations for Quantifying Wave Generation

- above equations represent a hybrid model containing:

  $\longrightarrow$ linear gravity waves

  $\longrightarrow$ Queney's topographic wake

  $\longrightarrow$ omega equation for QG vertical motion
- generation of gravity waves by balanced motions

In Closing

Topographic Waves at Small Rossby Number

- ▷ desingularized quadrature
- ▷ extension to 3D

Gravity Wave Generation at Moderate Rossby Number

- ▷ forcing of interior Queney problem by $\vec{Q}$-vector