Removable Circuits in Binary Matroids

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Abstract

We show that if M is a connected binary matroid of cogirth at least five which does not have both an F_7-minor and an F_7^*-minor, then M has a circuit C such that $M - C$ is connected and $r(M - C) = r(M)$.

1 Introduction

We shall consider the problem of finding sufficient conditions for the existence of a circuit in a given matroid M whose deletion leaves the rank or connectivity of M unchanged. The existence of such a circuit in graphs has been considered by various authors. The most general result for simple graphs can be deduced from a theorem of W. Mader [5, Satz 1].

Theorem 1 *Let k be a positive integer and G be a simple k-connected graph of minimum degree at least $k+2$. Then G has a circuit C such that $G - E(C)$ is k-connected.*

Stronger results for the special case when G is simple and $k = 2$ can be found in Jackson [4] and Thommassen and Toft [10].

It seems natural to ask if Theorem 1 can be extended to a graph G, which may contain multiple edges. We can obtain a partial result by applying Theorem 1 to the underlying simple graph of G, if G has no edges of multiplicity greater than two, and otherwise choosing C to be a 2-circuit of G belonging to an edge of multiplicity at least three, to deduce

Corollary 2 *Let k be a positive integer and G be a k-connected graph of minimum degree at least $k+3$. Then G has a circuit C such that $G-E(C)$ is k-connected.*

It follows from a result of Sinclair [9] that the bound $k+3$ in Corollary 2 can be reduced to $k+2$ for the special case when $k=1$. This is not true when $k=2$, however, as can be seen from an example constructed by N. Robertson and later B. Jackson (see [4]). However, replacing $k+3$ by $k+2$ when $k=2$ in Corollary 2 is valid for graphs which do not contain a vertex of degree four incident with two edge-disjoint 2-circuits by [9], for planar graphs by [1], and, more generally, graphs with no Petersen minor, by [2].

Oxley asked in [7, Problem 14.4.8] if the following partial extension of Theorem 1 when $k=2$ is valid for binary matroids: does every connected binary matroid of girth at least three and cogirth at least four have a circuit C such that $M-C$ is connected? L Lemos (see [2]) has constructed a cographic matroid of cogirth four which shows that the answer to Oxley's question is no. It remains an open problem, however, to decide if there exists an integer $t \geq 5$ such that all connected binary matroids M of cogirth at least t have a circuit C such that $M-C$ is connected. We shall show in Theorem 7 that this assertion is true with $t=5$ for binary matroids M which do not have both an F_7- and an F_7^*-minor. This gives a partial generalisation of Corollary 2 when $k=2$. Our proof uses the decomposition theory of Seymour in [8] which implies that a 3-connected, vertically 4-connected binary matroid which does not have both an F_7-minor and an F_7^*-minor is either graphic or cographic, or is isomorphic to R_{10}, F_7 or F_7^*. We shall first show that our result holds for graphic and cographic matroids. We then proceed by contradiction and show that a smallest counterexample to the result would be vertically 4-connected. It then only remains to check that the result holds for matroids obtained from R_{10}, F_7 or F_7^* by parallel extensions.

2 Graphs

We shall consider finite graphs which may contain multiple edges, but no loops. We consider a graph G to be 2-connected if $G - v$ is connected for all $v \in V(G)$. We shall use $E_G(v)$ to denote the set of edges of G incident with a vertex v and put $d_G(v) = |E_G(v)|$. We will suppress the subscript G when it is clear to which graph we are referring. Given a circuit C of G, put $|C| = |E(C)|$.

We first obtain, in Lemma 4 below, a slight extension of the case $k = 2$ of Corollary 2. We need this extension for our inductive proof on matroids. Lemma 4 itself follows from a result of Sinclair [9]. We include a proof in this paper for the sake of completeness. We shall use the following elementary result.

Lemma 3 *Let G be a graph on n vertices and C_0 be a circuit of G such that $|C_0| \leq 3$ and $n > |C_0|$. Suppose that for all $v \in V(G) - V(C_0)$ we have $d_G(v) \geq 4$. Then G has a circuit C such that $E(C_0) \cap E(C) = \emptyset$.*

Proof. If G is not 2-connected then choosing C to be any circuit in an end-block of G which does not contain C_0 we have $E(C_0) \cap E(C) = \emptyset$. Hence we may suppose that G is 2-connected.

Let $H = G - E(C_0)$. Suppose H is a forest. Then $|E(H)| \leq n - 1$. Let t be the number of edges between $V(C_0)$ and $V(G) - V(C_0)$. Then $|E(H)| = \frac{1}{2}(t + \sum_{v \in V(G)-V(C_0)} d_G(v))$. Since G is 2-connected, $t \geq 2$, and since $d(v) \geq 4$ for all $v \in V(G) - V(C_0)$, we have $|E(H)| \geq 2n - 2|C_0| + 1$. Thus $n \leq 2|C_0| - 2$. Since $n \geq |C_0|$, we have $|C_0| = 3$, and $n = 4$. Let $V(G) - V(C_0) = \{v\}$. Using the assumption that H is a forest, we have $d_G(v) \leq 3$. This contradicts an hypothesis on G and so the assumption that H is a forest must be false. ∎

Lemma 4 *Let G be a 2-connected graph on n vertices and C_0 be a circuit of G such that $|C_0| \leq 3$ and $n > |C_0|$. Suppose that for all $v \in V(G) - V(C_0)$ we have $d_G(v) \geq 5$. Then $G - E(C_0)$ has a circuit C such that $G - E(C)$ is 2-connected.*

Proof. Suppose the theorem is false and let G be a counterexample. By Lemma 3, we can choose a circuit C in $G - E(C_0)$. Let $H = G - E(C)$, let B_0 be the block of H which contains C_0 and B be an end-block of H

distinct from B_0. We may suppose that C has been chosen such that $|E(B)|$ is minimal. Let e be an edge of B chosen such that, if B contains a cut-vertex x of H, then e is incident with x. Since $d_G(v) \geq 5$ for all $v \in V(G) - V(C_0)$, at most one vertex of $B - e$ has degree less than two. Thus we may choose a circuit C' contained in $B - e$. Using the minimality of $|E(B)|$ and the fact that G is 2-connected we see that each end-block of $H - E(C')$ is incident with C and each component of $H - E(C')$ is incident with at least two vertices of C. Thus $G - E(C') = (H - E(C')) \cup E(C)$ is 2-connected. This contradicts the choice of G as a counterexample to the theorem. ∎

Given a graph G and $U \subseteq V(G)$, we use $N_G(U)$ to denote the set of vertices of $V(G) - U$ adjacent to a vertex of U and $G[U]$ to denote the subgraph of G induced by U. For $S \subseteq E(G)$, let G/S be the graph obtained from G by contracting the edges in S, and $V(S)$ the set of vertices of G incident with S.

We next show, in Lemma 6 below, that the case $k = 2$ of Corollary 2 can be extended to cographic matroids. We shall use the following elementary result.

Lemma 5 *Let G be a connected graph on n vertices and X_0 be a cocircuit of G such that $|X_0| \leq 3$ and $|E(G)| \geq n + |X_0| - 1$. Suppose that $G - X_0$ has girth at least four. Then $V(G) \neq V(X_0)$.*

Proof. Let H_1 and H_2 be the two components of $G - X_0$. Suppose $V(G) = V(X_0)$. Then $|V(H_i)| \leq 3$ and since $G - X_0$ has girth at least four, H_i is a tree for $1 \leq i \leq 2$. Thus

$$|E(G)| = |V(H_1)| - 1 + |V(H_2)| - 1 + |X_0| = n + |X_0| - 2.$$

This contradicts the hypothesis on $|E(G)|$. ∎

Lemma 6 *Let G be a 2-connected graph on n vertices and X_0 be a cocircuit of G such that $|X_0| \leq 3$ and $|E(G)| \geq n + |X_0| - 1$. Suppose that $G - X_0$ has girth at least five. Then there exists $v \in V(G) - V(X_0)$ such that $G/E(v)$ is 2-connected.*

Proof. Suppose the theorem is false and let G be a counterexample. By Lemma 5, we can choose a vertex v in $V(G) - V(X_0)$. Let $H = G/E(v)$ and

x be the vertex of H corresponding to $N_G(v) \cup \{v\}$. Then x is the unique cut vertex of H. Since $X_0 \cap E(v) = \emptyset$, X_0 is a cocircuit of H and hence is contained in a block B of H. Let $U = V(B) - x$. We may suppose that v has been chosen such that $|U|$ is maximal. Note that $N_G(U) \subseteq \{v\} \cup N_G(v)$. Furthermore, since G is 2-connected, $|N_G(U)| \geq 2$ and $G[U \cup N_G(U) \cup \{v\}]$ is 2-connected. Choose $v' \in V(H) - V(B)$. Then $v' \in V(G) - V(X_0)$. Let $H' = G/E(v')$ and x' be the vertex of H corresponding to $N_G(v') \cup \{v'\}$. Let B' be the block of H' containing X_0 and $U' = V(B') - x'$. Then $U \cup (N_G(U) - N_G(v'))$ is properly contained in $V(B')$. By the maximality of $|U|$ we must have $N_G(U) \subseteq N_G(v')$. Now the facts that $N_G(U) \subseteq \{v\} \cup N_G(v)$ and $|N_G(U)| \geq 2$ imply that $E(v) \cup E(v')$ contains a circuit of G of length at most four. This contradicts the fact that $G - X_0$ has girth at least five. ∎

3 Binary Matroids

We shall use the following operation on binary matroids from Seymour [8]. Given binary matroids M_1 and M_2 let $M_1 \triangle M_2$ be the binary matroid with $E(M) = E(M_1) \triangle E(M_2)$ and circuits all minimal non-empty subsets of $E(M)$ of the form $C_1 \triangle C_2$, where C_i is a circuit of M_i. We refer the reader to [7] for other definitions on matroids. Our main result is

Theorem 7 *Let M be a connected binary matroid which does not have both an F_7-minor and an F_7^*-minor. Let C_0 be a circuit of M such that $|C_0| \leq 3$ and $r(M) > r(C_0)$. Suppose $|X| \geq 5$ for all cocircuits X of M such that $X \cap C_0 = \emptyset$. Then $M - C_0$ has a circuit C such that $M - C$ is connected and $r(M - C) = r(M)$.*

Proof. We proceed by contradiction. Suppose the theorem is false and let M be a counterexample chosen such that $r(M)$ is as small as possible.

Claim 1 *M is vertically 3-connected.*

Proof. Suppose that M has a vertical 2-separation (S_1, S_2). Choose (S_1, S_2) such that $|S_1 \cap C_0|$ is minimal. Since $r(S_i) \geq 2$ we have $|S_i| \geq 2$. By [8, 2.6], $M = M_1 \triangle M_2$ for minors M_1 and M_2 of M such that $2 \leq r(M_i) < r(M)$, $E(M_1) \cap E(M_2) = C_0'$ for some 2-circuit $C_0' = \{f, g\}$ of M_i, and $E(M_i) - C_0' = S_i$ for $1 \leq i \leq 2$. Since M is connected each M_i is connected. Since M_i is a minor of M, M_i is binary and does not have both an F_7-minor and an F_7^*-minor. Since $C_0' \cap E(M) = \emptyset$ we have $C_0' \cap C_0 = \emptyset$. Since $|C_0| \le 3$, $|C_0 \cap E(M_1)| \le 1$.

Suppose $C_0 \cap E(M_1) = \{e\}$. Then $C_0 = C_1 \triangle C_2$ for some circuits C_i of M_i, $1 \le i \le 2$. Thus $|C_1| = 2$ and e is parallel to f and g in M_1. Let $h \in S_1 - e$ and Y be a circuit of M which meets both S_1 and S_2. Then $Y = Y_1 \triangle Y_2 \triangle$ for some circuits Y_i of M_i such that $|Y_i \cap C_0'| = 1$, $1 \le i \le 2$. Thus $Y_1 - C_0' + e$ is a circuit of both M_1 and M, and $r(S_1 - e) = r(S_1) \ge 2$. Similarly since $e \in C_0 \subseteq S_2 + e$ we have $r(S_2 + e) = r(S_2) \ge 2$. Thus $(S_1 - e, S_2 + e)$ is a vertical 2-separation of M. This contradicts the minimality of $|S_1 \cap C_0|$. Hence we must have $C_0 \cap E(M_1) = \emptyset$.

Let X_1 be a cocircuit of M_1 such that $X_1 \cap C_0' = \emptyset$. Then X_1 is a cocircuit of M such that $X_1 \cap C_0 = \emptyset$ so by an hypothesis of the theorem we have $|X_1| \ge 5$. Using the minimality of $r(M)$ we deduce that $M_1 - C_0'$ has a circuit C such that $M_1 - C$ is connected and $r(M_1 - C) = r(M_1)$. Since $M - C = (M_1 - C) \triangle M_2$ we have C is a circuit of $M - C_0$ such that $M - C$ is connected and $r(M - C) = r(M)$. This contradicts the choice of M. Thus M has no vertical 2-separation and hence M is vertically 3-connected. ∎

Claim 2 *M is vertically 4-connected.*

Proof. Suppose that M has a vertical 3-separation (S_1, S_2). Choose (S_1, S_2) such that $|S_1 \cap C_0|$ is minimal. Since $|C_0| \le 3$, $|C_0 \cap E(M_1)| \le 1$. We first show that $|S_i| \ge 4$ for $1 \le i \le 2$.

Suppose $|S_i| = 3$ for some $i \in \{1, 2\}$. Since $r(S_i) \ge 3$ we must have $r(S_i) = 3$. Since $r(S_1) + r(S_2) - r(M) = 2$ we have $r(S_j) = r(M) - 1$, for $j = 3 - i$. Thus the closure of S_j is a hyperplane of M. The complement of this hyperplane will be a cocircuit X_0 of M contained in S_i. Since $|X_0| \le |S_i| = 3$, it follows from an hypothesis of the theorem that $X_0 \cap C_0 \ne \emptyset$. Since M is binary we must have $|X_0 \cap C_0| = 2$. Since S_i is independent we must have $|C_0| = 3$ and $|S_j \cap C_0| = 1$. By the minimality of $|S_1 \cap C_0|$, we must have $i = 2$. Choosing $e_0 \in S_1 \cap C_0$ we have $r(S_1 - e) \le r(S_1)$ and, since $e_0 \in C_0 \subseteq S_2 + e_0$, $r(S_2 + e_0) = r(S_2) = 3$. Thus $(S_1 - e_0, S_2 + e_0)$ is either a vertical 2-separation of M, contradicting Claim 1, or it is a vertical 3-separation of M, contradicting the minimality of $|S_1 \cap C_0|$. Thus $|S_i| \ge 4$ for $i \in \{1, 2\}$.

By [8, 2.9], $M = M_1 \triangle M_2$ for minors M_1 and M_2 of M such that $3 \le r(M_i) < r(M)$, $E(M_1) \cap E(M_2) = C_0'$ for some 3-circuit $C_0' = \{f, g, h\}$ of

M_i, and $E(M_i) - C_0' = S_i$ for $1 \leq i \leq 2$. Since M is connected, each M_i is connected. Since M_i is a minor of M, M_i is binary and does not have both an F_7- and an F_7^*-minor. Since $C_0' \cap E(M) = \emptyset$ we have $C_0' \cap C_0 = \emptyset$.

Suppose $e \in C_0 \cap E(M_1)$. Then $C_0 = C_1 \triangle C_2$ for some circuit C_i of M_i, $1 \leq i \leq 2$. Thus $C_1 - C_0' = \{e\}$ and $1 \leq |C_1 \cap C_0'| \leq 2$. If $|C_1 \cap C_0'| = 2$ then replacing C_1 by $C_1' = C_1 \triangle C_0'$ we have $|C_1' \cap C_0'| = 1$. Thus we may assume without loss of generality that e is parallel to f in M_1. Let M_1' be the simple matroid obtained by replacing all parallel classes of M_1 by single elements and let f, g and h represent their own parallel classes in M_1'. Using Claim 1 it follows that M_1' is 3-connected. If f is a coloop $M_1' - \{g, h\}$ then C_0' would contain a cocircuit of M_1'. Since M_1' is binary this cocircuit would have size two and hence would contradict the fact that M_1' is 3-connected. Thus f is contained in some circuit of $M_1' - \{g, h\}$. Since e is parallel to f we deduce that M has a circuit which contains e and is contained in S_1. Hence $r(S_1 - e) = r(S_1) \geq 2$. Similarly since $e \in C_0 \subseteq S_2 + e$ we have $r(S_2 + e) = r(S_2) \geq 2$. Thus $(S_1 - e, S_2 + e)$ is a vertical 3-separation of M which contradicts the minimality of $|S_1 \cap C_0|$. Hence we must have $C_0 \cap E(M_1) = \emptyset$.

Let X_1 be a cocircuit of M_1 such that $X_1 \cap C_0' = \emptyset$. Then X_1 is a cocircuit of M such that $X_1 \cap C_0 = \emptyset$ so by an hypothesis of the theorem we have $|X_1| \geq 5$. Using the minimality of $r(M)$ we deduce that $M_1 - C_0'$ has a circuit C such that $M_1 - C$ is connected and $r(M_1 - C) = r(M_1)$. Since $M - C = (M_1 - C) \triangle M_2$ it follows that C is a circuit of M such that $M - C$ is connected and $r(M - C) = r(M)$. This contradicts the choice of M. Thus M has no vertical 3-separation and hence M is vertically 4-connected. ∎

We are now ready to complete the proof of the theorem. Let M' be the simple matroid obtained by replacing all parallel classes of M by single elements. By Claims 1 and 2, M' is a 3-connected vertically 4-connected binary matroid. By [8, 7.6 and 14.3], M' is either graphic or cographic, or is isomorphic to R_{10}, F_7 or F_7^*. Thus M is either graphic or cographic, or can be obtained from R_{10}, F_7 or F_7^* by a sequence of parallel extensions. If the latter alternative holds then since R_{10}, F_7 and F_7^* have many cocircuits of size four, $M - C_0$ must contain a circuit C of size two. The 3-connectivity of M' now implies that $M - C$ is connected and $r(M - C) = r(M)$. Hence M is graphic or cographic. Lemmas 4 and 6 now give a contradiction to the choice of M as a counterexample to the theorem. ∎

4 Closing Remarks

Remark 1 It follows from Corollary 2 that every connected graph G of minimum degree at least three has a circuit C such that $G - E(C)$ is connected. Thus every graphic matroid M of cogirth at least three has a circuit C such that $r(M) = r(M - C)$. The same result holds for a cographic matroid M of cogirth at least three. (This can be seen by considering the graph G for which M is the cographic matroid. Then G has girth at least three and the set of edges incident with any non-cutvertex of G will give the required circuit C of M.) The result does not extend to regular matroids of cogirth at least three since it does not hold for R_{10} (which has cogirth four). However, if M is a binary matroid which does not have both an F_7- and an F_7^*-minor, and has cogirth at least five, then we may apply Theorem 7 to a component of M to deduce that M has a circuit C such that $r(M) = r(M - C)$.

One may hope that all binary matroids M of sufficiently high girth have a circuit C such that $r(M) = r(M - C)$. This is not the case. To see this note that $r(M) = r(M-C)$ if and only if C does not contain any cocircuit of M. Thus, if M is identically self-dual (and in particular if $M = R_{10}$) then no such circuit can exist. The assertion now follows since there exist identically self-dual binary matroids of arbitrarily high cogirth. The column matroid of the parity check matrix of the binary Reed-Muller code $R(s, 2s+1)$, for example, is identically self dual and has cogirth 2^{s+1}.

Remark 2 It is not true that every connected matroid of sufficiently high girth has a circuit C such that $M - C$ is connected. This can be seen by considering the uniform matroid $U_{m,2m}$. It is still conceivable, however, that this may hold for binary matroids.

Problem 1 *Does there exist an integer t such that every connected binary matroid M of cogirth at least t has a circuit C such that $M - C$ is connected?*

Remark 3 We could also ask for sufficient conditions for the existence of a cocircuit in a matroid M the deletion of which preserves the connectivity of M. The following reult of P.D. Seymour (see [6, Lemma 6]) is in the spirit of this paper. It is a matroid analogue of an earlier graph theoretic result of Kaugars (see [3, p. 31]).

Lemma 8 *Let M be a connected binary matroid of girth and cogirth at least three. Then M has a cocircuit X such that $M - X$ is connected.*

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