Solution to Problem 10470 proposed by D. Knuth in the American Math. Monthly, 102 no. 7 (1995) p. 655

Luis Goddyn, Dept. of Mathematics and Statistics,
Simon Fraser University, Burnaby, BC, Canada V5A 1S6
goddyn@sfu.ca
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Answer to part (a). Let (a_{ij}) be an n by n matrix and let M be the set of permutations σ of $[n] := \{1, 2, \ldots, n\}$ such that $\prod_{i=1}^{n} a_{i,\sigma(i)} \neq 0$. Let $\sigma \in M$, and suppose that σ has exactly $z(\sigma)$ orbits on $[n]$. If (a_{ij}) is a special matrix, then for each orbit Z of σ, the multiset $\{a_{i,\sigma(i)} : i \in Z\}$ contains exactly one 1, with the remaining entries being -1. Thus $\prod_{i=1}^{n} a_{i,\sigma(i)} = (-1)^{n-z(\sigma)} = \text{sign}(\sigma)$ and we have the following.

$$\textit{For any special matrix } (a_{ij}),\ \det(a_{ij}) = \textstyle\sum_{\sigma\in M} \text{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} = |M|. \tag{1}$$

A subset $S \subseteq [n]$ is called a *barrier* of (a_{ij}) if $|N(S)| < |S|$ where $N(S) := \{j \in [n] : \exists i \in S, a_{ij} \neq 0\}$. Phillip Hall's theorem (On representatives of subsets. *J. London Math. Soc.* **10** (1935) 26–30) asserts the following.

For any matrix (a_{ij}), $|M| = 0$ if and only if (a_{ij}) has a barrier.

With this and (1) we have shown that a special matrix (a_{ij}) is minimal if and only if it has a barrier, but changing any entry a_{ij} with $i \geq j$ from 0 to 1 results in a matrix with no barriers.

Let (a_{ij}) be a special n by n matrix, and let $S \subseteq [n]$. Because of the -1 entries in (a_{ij}) we have $\{i+1 : i \in S - \{n\}\} \subseteq N(S)$ whence $|N(S)| \geq |S| - 1$. If S is a barrier, then this inequality is tight and we have the following.

$$\textit{Every barrier } S \textit{ of an } n \textit{ by } n \textit{ special matrix contains } n \textit{ and satisfies } N(S) = \{j : j-1 \in S - \{n\}\}. \tag{2}$$

Let (a_{ij}) be a minimal matrix and let S be a barrier of (a_{ij}) having minimum cardinality. Suppose there is an entry $a_{ij} = 0$ such that $j \leq i$. If either $i \notin S$ or $j \in N(S)$, then S is also a barrier of the special matrix obtained by changing a_{ij} to 1, contradicting the minimality of (a_{ij}). It follows that S is the unique barrier of (a_{ij}) and, by (2), the entries of (a_{ij}) are completely determined by S as follows.

$$a_{ij} = \begin{cases} 1 & \text{if } j \leq i \text{ and either } i \notin S \text{ or } j-1 \in S \\ -1 & \text{if } j = i+1 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

As (3) defines an n by n minimal matrix for any subset $S \subseteq [n]$ with $n \in S$, there are exactly 2^{n-1} such matrices.

Answer to part (b). Let $S = \{s_1, s_2, \ldots, s_k\} \subseteq [n]$ where $s_1 < s_2 < \cdots < s_k = n$, let $T = [n] - S$, and let (a_{ij}) be the minimal matrix determined by S as above. Each 1 which appears in (a_{ij}) has one of two types:

type-T: $a_{ij} = 1$ where $i \in T$ and $1 \leq j \leq i$.

type-S: $a_{ij} = 1$ where $i \in S$, $j - 1 \in S$ and $j \leq i$.

The number of type-T entries in (a_{ij}) is $\sum T$. We count the type-S entries by summing over the columns in $\{j : j-1 \in S\}$; for any $r \in \{1, 2, \ldots, k-1\}$, there are exactly $k - r$ type-S entries a_{ij} with $j = s_r + 1$, namely those with $i \in S \cap \{j, j+1, \ldots, n\} = \{s_{r+1}, s_{r+2}, \ldots, s_k\}$. In total there are $\sum_{r=1}^{k-1}(k-r) = \binom{|S|}{2}$ type-S entries in (a_{ij}). The number of zeros appearing on or below the diagonal in (a_{ij}) is calculated $\binom{n+1}{2} - \sum T - \binom{|S|}{2} = \sum S - \binom{|S|}{2} = \sum_{r=1}^{k}(s_r - r + 1)$. For a fixed k, this sum is maximized when $S = \{n-k+1, n-k+2, \ldots, n\}$ whence the sum equals $k(n-k+1)$. This expression attains the maximum value $\lfloor (n+1)^2/4 \rfloor$ when k is an integer closest to $(n+1)/2$. Including the zeros above the diagonal, we have that the maximum number of zeros in an n by n minimal matrix is

$$\lfloor (n+1)^2/4 \rfloor + \binom{n-1}{2} = \lfloor (3n^2 - 4n + 5)/4 \rfloor = n^2 - \lceil (n+5)(n-1)/4 \rceil.$$

□