NOWHERE-ZERO FLOWS IN REGULAR MATROIDS AND HADWIGER'S CONJECTURE

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To the memory of Reinhard Börger

Abstract. We present a tool that shows, that the existence of a k-nowhere-zero-flow is compatible with 1-,2- and 3-sums in regular matroids. As application we present a conjecture for regular matroids that is equivalent to Hadwiger's conjecture for graphs and Tuttes's 4- and 5-flow conjectures.

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1. Introduction

A (real) matrix is *totally unimodular (TUM)* if each subdeterminant belongs to $\{0, \pm 1\}$. Totally unimodular matrices enjoy several nice properties which give them a fundamental role in combinatorial optimization and matroid theory. In this note we prove that the TUM possesses an attractive property.

Let $S \subseteq \mathbb{R}$, and let A be a real matrix. A column vector f is a *S-flow of A* if $Af = 0$ and every entry of f is a member of $\pm S$.

For any additive abelian group Γ use the notation $\Gamma^* = \Gamma \setminus \{0\}$. For a TUM A and a column vector f with entries in Γ, the product Af is a well defined column vector with entries in Γ, by interpreting $(-1)\gamma$ to be the additive inverse of γ.

It is convenient to use the language of matroids. A regular oriented matroid M is an oriented matroid that is representable $M = M[A]$ by a TUM matrix A. Here the elements $E(M)$ of M label the columns of A. Each (signed) cocircuit $D = (D^+, D^-)$ of M corresponds to a $\{0, \pm 1\}$-valued vector in the row space of A and having minimal support. The $+1$-entries in this vector constitute the sets D^+. It is known [19, Prop. 1.2.5] that two TUMs represent the same oriented matroid if and only if the first TUM can be converted to the second TUM by a succession of the following operations: multiplying a row by -1, adding one row to another, deleting a row of zeros, and permuting columns (with their labels).

For $S \subseteq E(M)$ we use the notation $f(S) = \sum_{e \in S} f(e)$. Let $M = M[A]$ be the regular oriented matroid represented by the TUM A. Let $S \subseteq \Gamma$ where Γ is an abelian group. An *S-flow* of M is a function $f : E(M) \to S$ for which $Af = 0$, where f is interpreted to be a vector indexed by the column labels of A. For any $S \subseteq \Gamma$ we say that a regular matroid M has an S-flow if any of the TUMs that represent M has an S-flow. By the previous paragraph, this property of M is well defined. Since the rows of a TUM A generate the cocycle space of $M = M[A]$, we have that a function $f : E(M) \to \Gamma$ is a flow if and only if for every signed cocircuit $D = (D^+, D^-)$ we have that $f(D) = 0$ where $f(D)$ is defined to equal $f(D^+) - f(D^-)$.

Let Γ be a finite abelian group. Let M be a regular oriented matroid, and let $F \subseteq E(M)$ and let $f : F \to \Gamma$. Let $\tau_\Gamma(M, f)$ denote the number of Γ^*-flows of M which are extensions of f.

Theorem 1. *Let M be an regular oriented matroid. Let $F \subseteq E(M)$ and let $f, f' : F \to \Gamma$. Suppose that for every minor N of M satisfying $E(N) = F$, we have that f is a Γ-flow of N if and only f' is a Γ-flow of N. Then $\tau_\Gamma(M, f) = \tau_\Gamma(M, f')$.*

Proof. We proceed by induction on $d = |E \backslash F|$. If $d = 0$, then there is nothing to prove. Otherwise let $e \in E \backslash F$. If e is a coloop of M, then $\tau_\Gamma(M, f) = \tau_\Gamma(M, f') = 0$. If e is a loop of M, then by applying induction to $M \setminus e$, we have $\tau_\Gamma(M, f) = \tau_\Gamma(M, f') = (|\Gamma| - 1)\tau_\Gamma(M \backslash e, f)$. Otherwise we apply Tutte's deletion/contraction formula [3] and induction to get

$$\tau_\Gamma(M, f') = \tau_\Gamma(M/e, f') - \tau_\Gamma(M \backslash e, f') = \tau_\Gamma(M/e, f) - \tau_\Gamma(M \backslash e, f) = \tau_\Gamma(M, f).$$

□

Corollary 2. *Let D be a positively oriented cocircuit of a regular oriented matroid M. Let $f, f' : D \to \Gamma$. Suppose that for every $S \subseteq D$ we have that $f(S) = 0$ if and only if $f'(S) = 0$. Then $\tau_\Gamma(M, f) = \tau_\Gamma(M, f')$.*

Proof. Let N be a minor of M satisfying $E(N) = D$. Then $E(N)$ is a disjoint union $\bigcup_i D_i$ of positively oriented cocircuits of N [9, Prop. 9.3.1]. Thus f is a $\Gamma*$-flow of N if and only if f has no zeros, and $f(D_i) = 0$ for each i. The result follows from Theorem 1. □

Corollary 3. *Let M be a regular oriented matroid which has a Γ^*-flow f.*

1. *Let $e \in E(M)$ and $\gamma \in \Gamma^*$. Then M has a Γ^*-flow f' with $f'(e) = \gamma$.*
2. *Let D be a signed cocircuit of M of cardinality three. Let $f' : D \to \Gamma^*$ satisfy $f'(D) = 0$. Then f' extends to a Γ^*-flow of M.*

Proof. (1) In any minor N with $E(N) = \{e\}$, both f' and $f\restriction_{\{e\}}$ are Γ^*-flows of N if and only if N is a loop. Thus by Theorem 1 $\tau_\Gamma(M, f') = \tau_\Gamma(M, f) > 0$.

(2) Let $S \subset D$. For any $e \in D$ we have $f'(D \setminus \{e\}) = f'(D) - f'(e) = -f'(e) \neq 0$. Therefore $f'(S) = 0$ if and only if $S = D$. Since f is a Γ-flow and D is a positively oriented cocircuit of D we have $f(D) = 0$. Since $f(e) \neq 0$ for $e \in D$ we again have that $f(S) = 0$ if and only if $S = D$. It follows from Theorem 1 that $\tau_\Gamma(M, f') = \tau_\Gamma(M, f) > 0$.

□

A *k-nowhere zero flow (k-NZF)* of a regular oriented matroid M is an S-flow of M for $S = \{1, 2, \ldots, k-1\} \subset \mathbb{R}$. We frequently use the following observation of Tutte [15].

Proposition 4. *Let Γ be an abelian group of order k, and let $S = \{1, 2, \ldots, k-1\} \subset \mathbb{R}$. Then M has a k-NZF if and only if M has a Γ^*-flow. In particular, the existence of a Γ^*-flow in M depends only on $|\Gamma|$.*

A key step in the proof of Proposition 4 is the conversion of a Γ^*-flow into a k-NZF, where Γ is the group of integers modulo k. By modifying this argument, one can show that the statement of Corollary 3 remains true if each occurrence of the symbol Γ^* is replaced by the set of integers $S = \{\pm 1, \pm 2, \ldots, \pm(k-1)\}$. We omit the proof of this fact, as it is not needed in this paper.

2. Seymour decomposition

We provide here a description of Seymour's decomposition theorem for regular oriented matroids. We refer the reader to [13] for further details. We first describe three basic types of regular oriented matroids.

A oriented matroid is *graphic* if it can be represented by the $\{0, \pm 1\}$-valued vertex-edge incidence matrix of a directed graph, where loops and multiple edges are allowed. Any $\{0, \pm 1\}$-valued matrix which whose rows span the nullspace of a network matrix is called a *dual network matrix*. Dual network matrices are also TUM, and an oriented matroid is *cographic* is it is representable by a dual network matrix. The third class consists of all the all the orientations of one special regular matroid R_{10}. Every orientation of R_{10} can be represented by the matrix $[I|B]$ where B is obtained by negating a subset of the columns of the following matrix.

$$\begin{bmatrix} + & 0 & 0 & + & - \\ - & + & 0 & 0 & + \\ + & - & + & 0 & 0 \\ 0 & + & - & + & 0 \\ 0 & 0 & + & - & + \end{bmatrix} \tag{1}$$

Here "+" and "−" respectively denote $+1$ and -1.

Let M_1, M_2 be regular oriented matroids. If $E(M_1)$ and $E(M_2)$ are disjoint, then the *1-sum* $M_1 \oplus_1 M_2$ is just the direct sum of M_1 and M_2. The signed cocircuits of $M_1 \oplus_1 M_2$ are the signed subsets of $E(M_1) \cup E(M_2)$ which are signed cocircuits of either M_1 or M_2. If $M_1 \cap M_2 = \{e\}$ and e is neither a loop nor a coloop in each M_i, then the *2-sum* $M_1 \oplus_2 M_2$ has element set $E(M_1) \Delta E(M_2)$, where "Δ" is the symmetric difference operator. A signed cocircuit is a signed subset of $E(M_1 \oplus_2 M_2)$ that is either a signed cocircuit of M_1 or M_2, or is a signed set of the form

$$D = (D_1^+ \Delta D_2^+, D_1^- \Delta D_2^-) \tag{2}$$

where each (D_i^+, D_i^-) is a signed cocircuit of M_i, and $e \in (D_1^+ \cap D_2^+) \cup (D_1^- \cap D_2^-)$. If $M_1 \cap M_2 = B$ and $B = (B^+, B^-)$ is a signed cocircuit of cardinality 3 in each M_i, then the *3-sum* $M_1 \oplus_3 M_2$ has element set $E(M_1)\Delta E(M_2)$. A signed cocircuit is a signed subset of $E(M_1 \oplus_3 M_2)$ that is either a signed cocircuit of M_1 or M_2, or a signed subset of the form (2) where each (D_i^+, D_i^-) is a signed cocircuit of M_i, with $D_1 \cap D_2 = \emptyset$ and (B^+, B^-) equals one of the following ordered pairs:

$$((D_1^+ \cap B^+) \cup (D_2^+ \cap B^+) , \ (D_1^- \cap B^-) \cup (D_2^- \cap B^-))$$
$$((D_1^- \cap B^+) \cup (D_2^- \cap B^+) , \ (D_1^+ \cap B^-) \cup (D_2^+ \cap B^-)).$$

The oriented version of Seymour's decomposition theorem [13] and can be derived from [5, Theorem 6.6].

Theorem 5. *Every regular oriented matroid M can be constructed by means of repeated application of k-sums, $k = 1, 2, 3$, starting with oriented matroids, each of which is isomorphic to a minor of M and each of which is either graphic, cographic, or an orientation of R_{10}.*

We note that Schriver [12] states an equivalent version of Theorem 5 in terms of TUMs, that requires a second representation of R_{10} in (1) due to his implicit selection of a basis.

Here is the main tool of this paper, which we employ in two subsequent applications.

Theorem 6. *Let $k \geq 2$ be an integer and let $\mathcal{M}$ be a set of regular oriented matroids that is closed under minors. If every graphic and cographic member of $\mathcal{M}$ has a k-NZF, then every matroid in M has a k-NZF.*

Proof. Let $M \in \mathcal{M}$. We proceed by induction on $|E(M)|$. If M is an orientation of R_{10}, then M has a 2-NZF since R_{10} is a disjoint union of circuits, and each circuit is the support of a $\{0, \pm 1\}$-flow in M. If M is graphic or cographic, then we are done by assumption. Otherwise, by Theorem 5, M has two proper minors M_1, $M_2 \in \mathcal{M}$. such that $M = M_1 \oplus_i M_2$, for some $i = 1, 2, 3$. By induction, each M_i has a k-NZF. Thus by Proposition 4, both minors have a Γ^*-flow where Γ is any fixed group of order k. By Corollary 3, we may assume that these Γ^*-flows coincide on $M_1 \cap M_2$. Hence the union of these functions is a well defined Γ^*-flow on M and we are done by another application of Proposition 4. □

3. Tutte's flow Conjectures and Hadwiger's Conjecture

In this section we will present a conjecture that unifies two of Tutte's Flow Conjectures and Hadwiger's Conjecture on graph colorings.

Conjecture 7 (H(k)[4]). *If a simple graph is not k-colorable, then it must have a K_{k+1}-minor.*

While H(1) and H(2) are trivial, Hadwiger proved his conjecture for $k = 3$ and pointed out that Klaus Wagner proved that H(4) is equivalent to the Four Color Theorem [18, 2, 10]. Robertson, Seymour and Thomas [11] reduced H(5) to the Four Color Theorem. The conjecture remains open for $k \geq 6$.

Tutte [15] pointed out that the Four Color Theorem is equivalent to the statement that every planar graph admits an 4-NZ-flow. Generalizing this to arbitrary graphs he conjectured that

Conjecture 8 (Tutte's Flow Conjecture [15]). *There is a finite number $k \in \mathbb{N}$ such that every bridgeless graph admits a k-NZ-flow.*

and moreover that

Conjecture 9 (Tutte's Five Flow Conjecture [15]). *Every bridgeless graph admits a 5-NZ-flow.*

Note that the latter is best possible as the Petersen graph does not admit a 4-NZ-flow. Conjecture 8 has been proven independently by Kilpatrick [7] and Jaeger [6] with $k = 8$ and improved to $k = 6$ by Seymour [14].

Conjecture 9 has a sibling which is a more direct generalization of the Four Color Theorem.

Conjecture 10 (Tutte's Four Flow Conjecture [16, 17]). *Every graph without a Petersen-minor admits a 4-NZ-flow.*

In [16, 17] Tutte cited Hadwiger's conjecture as a motivating theme and pointed out that while

> "Hadwiger's conjecture asserts that the only irreducible chain-group which is graphic is the coboundary group of the complete 5-graph"

Conjecture 10 means that

"the only irreducible chain-group which is cographic is the cycle group of the Petersen graph."

The first statement refers to the case where the rows of a totally unimodular matrix A consist of a basis of signed characteristic vectors of cycles of a digraph.

Combining these we derive the following formulation in terms of regular matroids. First let us call any integer combination of the rows of A a *coflow.* Clearly, by duality resp. orthogonality, flows and coflows yield the same concept in regular matroids. Note that the existence of a k-NZ-coflow in a graph is equivalent to k-colorability [16].

Conjecture 11 (Tutte's Four Flow Conjecture, matroid version). *A regular matroid that does not admit a 4-NZ-flow has either a minor isomorphic to the cographic matroid of the K_5 or a minor isomorphic to the graphic matroid of the Petersen graph.*

Equivalently, we have

Conjecture 12 (Hadwigers's Conjecture for regular matroids and $k = 4$). *A regular matroid that is not 4-colorable, i.e. that does not admit a NZ-4-coflow, has a K_5 or a Petersen-dual as a minor.*

Some progress concerning this Conjecture was made by Lai, Li and Poon using the Four Color Theorem

Theorem 13 ([8]). *A regular matroid that is not 4-colorable has a K_5 or a K_5-dual as a minor.*

Tutte's Five Flow Conjecture now suggests the following matroid version of Hadwiger's conjecture:

Conjecture 14 (Hadwigers's Conjecture for regular matroids and $k \geq 5$). *If a regular matroid is not k-colorable for $k \geq 5$, then it must have a K_{k+1}-minor.*

Theorem 15. (1) *Conjecture 11 is equivalent to Conjecture 10.*
(2) *Conjecture 14 for $k = 5$ is equivalent to Conjecture 9.*
(3) *Conjecture 14 for $k \geq 6$ is equivalent to Conjecture 7.*

Proof. (1) By Weiske's Theorem [4] a graphic matroid has no K_5^*-minor. Hence Conjecture 11 clearly implies Conjecture 10. The other implication is proven by induction on $|E(M)|$. Consider a regular matroid M, that is not 4-colorable, i.e. that does not admit a NZ-4-coflow. Clearly, M cannot be isomorphic to R_{10}. If M is graphic, it must have a K_5-minor by the Four Color Theorem [2, 10] and an observation of Klaus Wagner [18]. If M is cographic it must have a Petersen-dual-minor by Conjecture 10. Otherwise, by Theorem 5, M has two proper minors M_1, $M_2 \in \mathcal{M}$. such that $M = M_1 \oplus_i M_2$, for some $i = 1, 2, 3$ and at least one of them is not 4-colorable by Theorem 6. Using induction we find either a Petersen-dual-minor or a K_5-minor in one of the M_i and hence also in M. Thus, Conjecture 10 implies Conjecture 11.
(2) We proceed as in the first case using $H(5)$ for graphs [11] instead of the Four Color Theorem.
(3) We proceed similar to the first case, with only a slight difference in the base case. If M is graphic, it must have a K_{k+1}-minor by Conjecture 7. M cannot be cographic by Seymour's 6-flow-theorem [14]. □

Remark 16. *James Oxley pointed that Theorem 15 could also be proven using splitting formulas for the Tutte polynomial (see e.g. [1]), Seymour's decomposition and the fact that the flow number as well as the chromatic number are determined by the smallest non-negative integer non-zero of certain evaluations of the Tutte polynomial.*

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