Hold'Em Bad-Beat Jackpot Computations

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Abstract

We determine the probabilities for the occurrences of bad-beat jackpots in hold'em under several qualification schemes. The probabilities are determined exactly under the assumption a qualifying semideal is completed.

1 Introduction

We are going to calculate the probability for a bad-beat qualifying semideal occurring in hold'em. It is important to emphasize that the bad-beat jackpot may not occur because the semideal may not play out to completion. In other words, either a player who would have had a qualifying hand folds before the qualifying hand is realized, or all players fold before the hand finishes and qualifying hands simply are not realized. Thus, the actual probability for the jackpot to be paid is smaller than the probability we are calculating. How much smaller is not a simple question.

We work with semideals rather than deals because we don't care which players qualify for a bad-beat jackpot, we care only that two or more players qualify. We use the term *semideal completion* to mean that certain cards are specified on board and in players' hands and then we complete the specified cards to a semideal. For example, we might specify the five cards comprising the board and one player's hand. A semideal completion is then the random choice of 18 cards to be dealt to the other nine players.

Here are some general counting results. The total number of semideals for 10 players is given by observing there are $\binom{52}{5}$ ways to chose five cards for the board, there are $\binom{47}{20}$ ways to choose 20 cards from the remaining 47 to go to the players, and there are 19!! ways to partition the 20 cards into hands of two cards each. Hence, the total number of semideals for 10 players is

$$\binom{52}{5}\binom{47}{20}19!! = 16,611,978,703,557,549,675,134,772,000. \qquad (1)$$

We also are going to want to count the number of semideal completions. If m cards have been specified, then there are $52-m$ left in the deck. If we have k players whose hands have not been specified, then we must choose $2k$ cards from

$52-m$ which can be done in $\binom{52-m}{2k}$ ways. The number of ways of splitting the $2k$ cards into k hands is $(2k-1)!!$. Thus, the number of semideal completions with m cards specified and k players whose hands are arbitrary is given by

$$\binom{52-m}{2k}(2k-1)!! \tag{2}$$

2 Four Eights Or Better

In this section we are going to use the qualifying conditions used by PartyPoker, namely, a player must lose with four eights or better. In addition, for a hand to qualify the following conditions must hold:

- a player must have a pair in her hand for four-of-a-kind;
- a player must use her best hand; and
- both hole cards must play for a straight flush.

Let's clarify the preceding rules with two examples. A player holding A-K with a board of J-Q-K-K-K has four kings with an ace kicker for her best hand so that both her hole cards play, but her hand does not qualify because she must have two kings in her hand. A player holding 7-8 of spades with 9-10-J-Q of spades on board cannot claim she has a straight flush to the J so that both her hole cards play making her hand qualify. Her best hand is a straight flush to the Q and this hand uses only one of her hole cards. Thus, her hand does not qualify.

From (1) we know the total number of semideals for 10 players. What we want to do is count the number of semideals that have two or more players with qualifying hands. The probability is calculated easily from these two numbers.

We are going to count qualifying semideals by considering boards. There are some boards that allow as many as four qualifying hands and there are boards that allow just two qualifying hands. The various patterns of what boards allow are given in Table 1 below.

four-of-a-kind	straight flush
2	2
2	1
2	0
1	2
1	1
0	2

Table 1

2.1 2 & 2

For a board to allow two four-of-a-kind hands and two straight flushes, it must have the form xxyyz, where the three ranks x,y,z are consecutive and allow straight flushes on each end, the ranks x and y are 8 or bigger, and there are three suited cards (hence, of ranks x, y, and z). The only possible rank sets are {7,8,9}, {8,9,10}, {9,10,J}, and {10,J,Q}.

The rank set {7,8,9} has 4 choices for the suit and 3 choices for each of the cards of ranks 8 and 9. This produces 36 boards. For each of the other three rank sets, there are 4 choices for the suit, 3 choices for which ranks will have pairs, and 3 choices for each pairing card. This gives 108 boards for each rank set.

Altogether we have 360 boards that allow two four-of-a-kinds and two straight flushes. We now want to determine how many semideals that have at least two qualifying hands arise for these boards.

Let N_4 denote the semideals with four qualifying hands. This means that four hands are specified leaving six arbitrary hands from 39 cards. Thus,

$$N_4 = 360\binom{39}{12}11!! = 14,634,986,164,999,200.$$

Let N_3 denote the number of incidences of semideals having at least three qualifying hands. For each board there are 4 ways of choosing 3 of the possible qualifiers. This then specifies 6 cards in hands leaving 41 cards for seven arbitrary hands. We obtain

$$N_3 = 4 \cdot 360\binom{41}{14}13!! = .6,857,536,374,456,768,000$$

Similarly, letting N_2 denote the incidences of semideals having at least two qualifying hands, we obtain

$$N_2 = 6 \cdot 360\binom{43}{16}15!! = 1,161,066,627,400,211,532,000.$$

We use inclusion-exclusion to count the number of semideals with at least two qualifying hands. We claim that

$$N_2 - 2N_3 + 3N_4 = 1,147,395,459,609,792,993,600 \qquad (3)$$

is the number of qualifying semideals. If a semideal has exactly two qualifying hands it is counted once in N_2 and in no other term. If a semideal has exactly three qualifying hands, it is counted three times in N_2 and twice in $2N_3$ so that it is counted once altogether. Likewise, a semideal with four qualifying hands is counted once. This completes the subcase of a board that allows four qualifying hands.

2.2 2 & 1

For a board to allow two four-of-a-kind hands and one straight flush, it again has the form xxyyz, where the three ranks allow only one straight flush and both x and y are at least 8. The rank sets here fall into four groupings.

The rank sets {5,8,9}, {6,8,10}, {6,9,10}, {7,9,J} and {7,10,J} have two big ranks and allow a straight flush in just one way. There are 4 choices for suit and 9 choices for the pairing cards. Altogether there are 180 boards for these 5 rank sets.

The rank sets {6,8,9}, {7,8,10} and {7,9,10} have two big ranks and allow a straight flush in one of two ways. There are four choices for suit and 9 choices for the pairing cards. Hence, there are 108 boards for these three rank sets.

Rank sets {8,9,Q}, {8,10,Q}, {8,J,Q}, {9,10,K}, {9,J,K}, {9,Q,K}, {10,J,A}, {10,Q,A}, {10,K,A}, {J,Q,A}, {J,K,A}, and {Q,K,A} have three big ranks and allow a straight flush in just one way. There are 4 choices for suit, 3 choices for which ranks are paired, and 9 choices for the pairs. This leads to 1,296 boards of this kind.

The rank sets {8,9,J}, {8,10,J}, {9,10,Q}, {9,J,Q}, {10,J,K} and {10,Q,K} have three big ranks and allow a straight flush in one of two ways. There are 4 choices of suit, 3 choices for the ranks of the pairs, and 9 choices for the pairs. This produces 648 boards of this kind.

We now want to determine how many semideals with at least two qualifying hands arise from these boards. Let N_3 be the number of semideals with three qualifying hands. We are specifying three hands leaving seven arbitrary hands to be chosen from 41 cards. Furthermore, the boards that allow straight flushes in two different ways have two different ways to specify three hands. Thus,

$$N_3 = 2988\binom{41}{14}13!! = 14,229,387,976,997,793,600.$$

Let N_2 be the incidences of semideals with at least two qualifying hands for the types of boards under consideration. For those that allow a straight flush in one way, there are 3 ways to choose two specified hands and for those that allow a straight flush in one of two ways, there are 5 ways to specify two hands. This yields

$$N_2 = ((3 \cdot 1476) + (5 \cdot 756))\binom{43}{16}15!! = 4,412,053,184,120,803,821,600.$$

The total number of semideals for these boards with at least two qualifying hands is given by

$$N_2 - 2N_3 = 4,383,594,408,166,808,234,400 \tag{4}$$

because any semideal with three qualifying hands is counted three times in the N_2 term.

2.3 2 & 0

We now consider boards that allow two players to have four-of-a-kind of ranks 8 and higher. The board again has the form xxyyz, where x and y are ranks of 8 and more and the board cannot allow a straight flush.

For any given xxyy, there are 36 ways to choose pairs of ranks x and y. For AA88, KK88 and AA99, the remaining card z may be any of 44. This gives $44 \cdot 108 = 4,752$ boards. There is not so much freedom for z in the remaining cases.

Of the 36 ways of choosing xxyy, 6 of them have the pairs in the same two suits, 24 of them have the pairs sharing one suit, and the other 6 have no suit in common. For example, if we have 8899 with two suits in common, then z cannot be any of {5,6,7,10,J,Q} in either of the two suits because the board would allow a straight flush. Hence, z is restricted to 32 cards. The following table contains the number of choices for the fifth card for all combinations of the possible ranks for the pairs xxyy.

x,y	2 suits	3 suits	4 suits
8-9,9-10,10-J	32	38	44
8-10,9-J,10-Q,J-Q	34	39	44
8-J,9-Q,10-K, J-K,Q-K	36	40	44
8-Q,9-K,10-A J-A,Q-A,K-A	38	41	44

TABLE 2

Using Table 2 and earlier information we calculate that there are 82,080 boards of the form under discussion in this subsection. The number of completions to semideals is then

$$82,080\binom{43}{16}15!! = 44,120,531,841,208,038,216,000. \qquad (5)$$

2.4 1 & 2

We move to boards allowing one four-of-a-kind and two straight flushes. There are more subcases and some care required to count boards accurately. We break the boards into two subclasses. The first subclass consists of the boards that allow two straight flushes with just three suited cards. This forces the three suited cards to have consecutive ranks. The possible rank sets are {3,4,5}, {4,5,6}, ..., {10,J,Q}. The rank sets {3,4,5}, {4,5,6}, and {5,6,7} require a big pair on board as well. The pair AA does not work for {3,4,5} because there cannot be a player with four aces and another with a straight flush using A2. Hence, there are 6 ranks for big pairs with {3,4,5}, 6 ranks for {4,5,6}, and 5 ranks for {5,6,7}. This gives $4(36 + 36 + 30) = 408$ boards for the three rank sets discussed.

The rank set {6,7,8} has more ways that allow four-of-a-kind. We can have either a pair of ranks J, Q, K or A, or we can have another 8 on board with a card z. The card z cannot be an 8 or any of the 4 cards needed for the straight flushes. Thus, z can be any of 42 cards. Thus, the number of boards for {6,7,8} is $4(24 + 126) = 600$.

In a similar way, we get $4(18+234) = 1,008$ boards for {7,8,9} because there are two big ranks in the set and we do not let the board allow two players to make four-of-a-kind. Proceeding in the same way, for the rank set {8,9,10} we get $4(12+324) = 1,344$ boards; for the rank set {9,10,J} we get $4(6+324) = 1,320$ boards; and for the rank set {10,J,Q} we get $4 \cdot 324 = 1,296$ boards.

The second subclass of boards are those that allow two straight flushes with four suited cards on board. Thus, we are interested in rank sets with four ranks. Note that in order for the board to allow four-of-a-kind there must be at least one big rank in the rank set.

A rank set with three small ranks and one big rank produces just 12 boards because there are 4 choices for suit and 3 choices for the pairing card. Here is a list of such rank sets: {A,4,5,7},{2,4,5,8},{2,5,6,8}, {2,5,6,9},{3,5,6,8},{3,5,6,9}, {3,6,7,9},{3,6,7,10},{4,6,7,9}, and {4,6,7,10}

A rank set with two small ranks and two big ranks produces 24 boards as is easily seen. The rank sets are: {A,4,5,8},{4,7,8,10}, {4,7,8,J},{5,7,8,10}, and {5,7,8,J}.

A rank set with one small rank and three big ranks gives rise to 36 boards. The rank sets are: {5,8,9,J},{5,8,9,Q},{6,8,9,J},{6,8,9,Q}, {6,9,10,Q},{6,9,10,K}, {7,9,10,Q},{7,9,10,K},{7,10,J,K}, and {7,10,J,A}.

A rank set with four big ranks gives rise to 48 boards. The rank sets are {8,10,J,K} and {8,10,J,A}.

We use inclusion-exclusion to count the number of semideal completions for the boards but there is a complication that arises for the first time in this subsection. We use the rank set {3,5,6,8} to illustrate the complication. Suppose the board has the 3,5,6,8 of spades and the 8 of clubs. This board supports three players having qualifying hands, but it supports an additional semideal with two players having qualifying hands, namely, one player holding 8-8 and another player holding 4-7 of spades. We have to take this into account when we do the counting.

The total number of boards described above is 6,672. Each has a unique way of producing three qualifying hands. Letting N_3 be the number of semideals with three qualifying hands leads to

$$N_3 = 6,672 \binom{41}{14} 13!! = 31,773,251,868,316,358,400.$$

In order to determine the number N_2 of incidences of two qualifying hands, we have to know how many boards allow an extra semideal with two qualifying hands. All the boards with three consecutive suited cards allow the extra semideal with two qualifying hands. Of the rank sets with four suited cards, only {A,4,5,8},{2,5,6,9},{3,6,7,10}, {4,7,8,J},{5,8,9,Q},{6,9,10,K}, and

{7,10,J,A} produce boards that do not have the extra qualifying situation. This amounts to 180 boards that do not have the extra semideal.

Let N_2 be the incidence of semideals with at least two qualifying hands. We have that

$$N_2 = (4 \cdot 6492) + (3 \cdot 180) = 14,248,867,666,261,484,856,600.$$

Using inclusion-exclusion we obtain

$$N_2 - 2N_3 = 14,185,321,162,524,852,139,800 \qquad (6)$$

semideals with at least two qualifying hands for the boards in this subsection.

2.5 1 & 1

We now consider boards that allow one four-of-a-kind and one straight flush. For the board to allow a straight flush, there must be three suited ranks for which two cards make a straight flush. We look at boards from this standpoint. There are five patterns the rank sets producing the straight flush can have: {x,x+1,x+3}, {x,x+1,x+4}, {x,x+2,x+3}, {x,x+2,x+4}, and {x,x+3,x+4}. We'll analyze each pattern separately.

The rank sets {x,x+1,x+3} allow only one straight flush at a time as x runs from A up through J. When x = A and x = J, there is just one way to make a straight flush, whereas, for all other values of x, there are two ways to make a straight flush. There are 4 choices for the suit in all of the rank sets. Because the board also must allow a big pair, there are completions to boards in different ways. Let's examine them.

When x = A, the board may be completed with any pair whose rank is chosen from 8 through K. There are 36 such pairs. The other possibility is to choose an A and some card from the 45 that are legal. This gives us $4(36 + 135) = 684$ boards when x = A.

When x = 2, 3 or 4, there are 4 choices for suit and the board must be completed with a big pair. There are 42 possible pairs but there is a subtle catch. If the big pair is A-A, then there cannot be both quad aces and a straight flush with A-4 when x = 2. So some of the boards allow two qualifying hands in two ways and others allow two qualifying hands in just one way. That becomes important when we perform the inclusion-exclusion calculation.

The next table gives the number of boards for the five types of rank sets and distinct values of x. The numbers in the table give the number of boards for the five types of rank sets. The numbers in parentheses give the number of those boards that allow two qualifying hands in just one way.

x	x,x+1,x+3	x,x+1,x+4	x,x+2,x+3	x,x+2,x+4	x,x+3,x+4
A	684(684)	672(672)	672(666)	672(648)	612(612)
2	168(24)	168(168)	144(24)	156(144)	132(132)
3	156(12)	156(156)	132(0)	156(144)	132(132)
4	156(24)	624(624)	132(0)	656(608)	588(588)
5	648(36)	612(612)	588(12)	624(576)	972(972)
6	624(36)	588(588)	972(12)	1032(984)	948(948)
7	1032(48)	984(984)	948(12)	1008(960)	924(924)
8	1368(60)	1308(1308)	1260(12)	1344(1248)	1380(1284)
9	1356(72)	1296(1296)	1392(60)	1332(1332)	1368(1368)
10	1344(48)	1284(1284)	1380(48)	1320(1320)	1356(1356)
J	1356(1356)	-	1356(1356)	-	-

TABLE 3

The total number of boards in Table 3 is 42,272. Of these boards, 28,574 allow two qualifying hands in one way and the remaining 13,698 allow two qualifying hands in two ways.

There are a few more boards that allow one four-of-a-kind and one straight flush that have not been included. A board of the form 5-6-7-8-8 with four suited cards does not allow two qualifying straight flushes because only a player holding 9-10 of the suit qualifies as all other straight flushes use four cards from the board. We get 12 boards from the rank set {5,6,7,8}, 24 boards from {6,7,8,9}, 36 boards from {7,8,9,10}, and 48 boards from each of {8,9,10,J}, {9,10,J,Q}, {10,J,Q,K}, and {J,Q,K,A}. This gives us another 264 boards allowing two qualifying hands in one way.

The total number of board completions to qualifying semideals is then $28,574+264+2(13698)=56,234$. Thus, the number of qualifying semideals we obtain is

$$56,234\binom{43}{16}15!!=30,227,509,595,010,877,449,300. \qquad (7)$$

2.6 0 & 2 plus some hybrids

We now want to determine the boards that allow two straight flushes but no four-of-a-kinds, or two straight flushes and a four-of-a-kind but not all three simultaneously (we are calling the latter hybrids). We partition the boards according to how the straight flushes are allowed.

The first type we consider are the boards with three suited cards of consecutive ranks starting with {3,4,5} through {10,J,Q}. The interesting part of the counting is determining how many ways the three suited cards can be completed to a board. We shall cover the details for {3,4,5} to illustrate the process.

We cannot choose the A or 2 in the suit of the 3-4-5 or else two straight flushes are not allowed. We cannot choose the 6 in the suit because a player holding A-2 would not have a qualifying hand because her best hand would use four cards from the board. Hence, we are choosing 2 cards from 46 to complete the board. This can be done in 1,035 ways. However, the board cannot allow four-of-a-kind unless it kills one of the straight flushes (a hybrid board). So we remove the 36 pairs of ranks 8 through K. We remove the three A-A pairs involving the A of the suit, but we do not remove the other A-A pairs because these boards allow either two straight flushes or a straight flush and a four-of-a-kind. Therefore, we have 996 board completions. Multiplying by 4 yields 3,984 boards for the rank set {3,4,5}.

There is another point to mention regarding the hybrid boards in the preceding paragraph. A board of A-A-3-4-5, with the 3-4-5 suited, allows two qualifying hands in three different ways: Either two straight flushes, or four aces and a 2-3-4-5-6 straight flush, or four aces and a 3-4-5-6-7 straight flush.

The rank set {4,5,6} behaves quite similarly except that rank 8 gives us 12 hybrid boards with 8-8 pairs. So we again have 3,984 boards of which 12 are hybrids. These 12 hybrids also allow two qualifying hands in three different ways.

The rank set {5,6,7} starts by choosing 2 cards from 46 in 1,035 ways. We exclude pairs of ranks 10 through A. We also remove 8-8 and 9-9 pairs that include a card of the 5-6-7 suit. The other 8-8 and 9-9 pairs give us hybrid boards. Hence, there are 3,996 boards of which 12 are hybrid boards allowing two qualifying hands in two ways, and 12 hybrid boards that allow two qualifying hands in three ways.

The rank set {6,7,8} behaves somewhat differently because now we cannot choose anymore 8s. Hence, we are excluding 9 cards so that we are choosing 2 cards from 43 to start. We remove the 24 pairs with ranks J through A and we remove the 3 pairs of rank 10 that include a card of the same suit. This gives us $4 \cdot 876 = 3,504$ boards of which 12 are hybrids with two ways of producing two qualifying hands (the 12 boards involving 9-9) and 12 are hybrids with three ways of producing two qualifying hands.

Continuing in the vein started by {6,7,8}, we obtain 3,024 boards for {7,8,9} with the usual 24 hybrid boards, 2,604 boards for {8,9,10} with 24 hybrid boards, 2,624 boards for {9,10,J} (note that the pair AK in the suit also must be excluded and now hybrid 8-8 pairs come into play) with 36 hybrid boards, and 2,520 boards for {10,J,Q} of which 48 are hybrid boards.

The sum of the above boards is 26,240 boards of which 108 allow two qualifying hands in three ways, 96 allow them in two ways, and the remaining 26,036 allow them in just one way.

The second type we consider are the boards with four suited cards of ranks that allow two straight flushes. These rank sets have the forms {x,x+2,x+3,x+5}, {x,x+2,x+3,x+6}, {x,x+3,x+4,x+6} and {x,x+3,x+4,x+7}. Eliminating the possibility of four-of-a-kind is easier here because all we have to do is make certain the fifth card does not pair any big card among the four suited cards. The next table contains the information.

x	x,x+2,x+3,x+5	x,x+2,x+3,x+6	x,x+3,x+4,x+6	x,x+3,x+4,x+7
A	-	-	180	164
2	180	164	168	164
3	168	164	168	164
4	168	164	156	152
5	156	152	144	140
6	144	140	144	140
7	144	140	144	140
8	132	128	-	-

TABLE 4

The sum of the entries in the table is 4,312 and each yields two qualifying hands in only one way.

There are rank sets with 5 ranks such that if all five cards are in the same suit, then the boards allow two straight flushes. If the middle rank is x, then it is the largest rank in one straight flush and the smallest rank in another straight flush. If either x-1 or x+1 is in the rank set, then we have four ranks that give the two straight flushes and the board was counted in the last subcase. So both x-1 and x+1 are missing from the rank set. There are then 3 ways to put the smaller and larger ranks on each side of x giving us 9 rank sets for each middle rank x. The middle rank varies from 5 up to 10. Hence, there are 54 rank sets which yields 216 boards of this last type.

Summing all the boards with possible numbers of completions to two qualifying hands gives 31,080. This then leads to

$$31,080\binom{43}{16}15!! = 16,706,458,694,258,599,266,000 \qquad (8)$$

qualifying semideals.

2.7 Conclusion For Four Eights Or Better

We take the sum of (3), (4), (5), (6), (7) and (8) to obtain

$$110,770,811,160,778,968,299,100$$

semideals that produce at least two qualifying hands for the bad-beat jackpot. Dividing by (1) gives a probability of .00000667 that a bad beat semideal would occur. This is approximately 1/150,000.